14.451 Lecture Notes 8

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1 Stochastic dynamic programming: an example

We now turn to analyze problems with uncertainty, in discrete time. We begin with an example that illustrates the power of recursive methods.

Take an unemployed worker with linear utility function. The worker is drawing wage-offers from a known distribution with continuous c.d.f. $F(w)$ on $[0, \bar{w}]$. At any point in time, he can stop and accept the offer. If he accepts he gets to work at wage w and then works forever getting utility $w/(1-\beta)$.

Sequence setup: history is sequence of observed offers

$$w^t = (w_0, w_1, ..., w_t).$$

A *plan* is to stop or not after any possible history, i.e., choose $\chi(w^t) \in \{0, 1\}$. The stopping time T is a random variable that depends on the plan $\chi(.)$: T is the first time where $\chi(w^t) = 1$. Objective is to choose $\chi(.)$ to maximize

$$E\left[\frac{\beta^T}{1-\beta} w_T\right].$$

Recursive setup. State variable: did you stop in the past? if yes what wage did you accept? So the state space is now $X = \{unemployed\} \cup R_+$. The value after stopping at wage w is just $V(w) = w/(1-\beta)$. So we need to characterize $V(unemployed)$, which we will denote V^U.

Each period decision after *never stopped* is

$$\max\left\{\frac{w}{1-\beta}, \beta V^U\right\}$$

or, equivalently,

$$\max_{\chi \in \{0,1\}} \chi \frac{w}{1-\beta} + (1-\chi)\beta V^U.$$

So optimal policy is to stop if $w > \hat{w}$, not stop if $w < \hat{w}$, and indifference if $w = \hat{w}$, where $\hat{w} = (1-\beta)\beta V^U$.

Bellman equation

$$V^U = \int_0^{\bar{w}} \max\left\{\frac{w}{1-\beta}, \beta V^U\right\} dF(w).$$

We can rewrite it in terms of the cutoff $\hat{w}$ and we have

$$\begin{aligned}\hat{w} &= (1-\beta)\beta\int_0^{\bar{w}} \max\left\{\frac{w}{1-\beta}, \beta V^U\right\} dF(w) \\ &= \beta\int_0^{\bar{w}} \max\{w, \hat{w}\}\, dF(w)\end{aligned}$$

find fixed point, here simply find $\hat{w}$ that solves

$$\hat{w} = T(\hat{w})$$

where

$$T(v) \equiv \beta\int_0^{\bar{w}} \max\{w, v\}\, dF(w).$$

Properties of this map:

- it is continuous increasing on $[0, \bar{w}]$;
- has derivative

  $$T'(v) = \beta F(v) - \beta v f(v) + \beta v f(v) = \beta F(v) \in [0, \beta]$$

  for $v \in (0, \bar{w})$ (here we use continuous distribution);
- has $T(0) = E[w]$ and $T(\bar{w}) = \bar{w}$.

Therefore, a unique fixed point $\hat{w}$ exists and is in $(0, \bar{w})$ (you can use contraction mapping to prove it).

Comparative statics 1. An increase in β increases the cutoff $\hat{w}$.

Just look at

$$T(v) = \beta\int_0^{\bar{w}} \max\{w, v\}\, dF(w)$$

and see that it is increasing in both β and v at the fixed point $\hat{w}$.

Comparative statics 2. A first-order stochastic shift in the distribution F leads to a (weak) increase in $\hat{w}$.

Comparative statics 3. A second-order stochastic shift in the distribution F leads to a (weak) increase in $\hat{w}$.

What is first-order and second-order stochastic dominance? Take two distributions F and G on R.

Definition 1 *The distribution F dominates the distribution G in the sense of 1st order stochastic dominance iff*

$$\int h(x)\, dF(x) \geq \int h(x)\, dG(x)$$

for all monotone functions $h : R \to R$.

Definition 2 *The distribution F dominates the distribution G in the sense of 2nd order stochastic dominance iff*

$$\int h(x)\, dF(x) \geq \int h(x)\, dG(x)$$

for all convex functions $h : R \to R$.

Sometimes you see stochastic dominance (1st and 2nd order) defined in terms of comparisons of the c.d.f. of F and G and then the definitions above are theorems!

Exercise: using the definitions above prove comparative statics 2 and 3.

Characterizing the dynamics. Let us make the problem more interesting (and stationary) by assuming that when employed agents lose their job with exogenous probability λ. The state space is still $X = \{\textit{unemployed}\} \cup R_+$.

Now the Bellman equation(s) are

$$\begin{aligned} V(w) &= w + \beta\left(\lambda V^U + (1-\lambda) V(w)\right) \\ V^U &= \int \max\left\{V(w), \beta V^U\right\} dF(w) \end{aligned}$$

From the first we get

$$V(w) = \frac{w + \beta\lambda V^U}{1 - \beta(1-\lambda)}$$

and we have to find V^U from

$$V^U = \int \max\left\{\frac{w + \beta\lambda V^U}{1 - \beta(1-\lambda)}, \beta V^U\right\} dF(w)$$

Exercise: prove that this defines a contraction with modulus β.

So we still have a cutoff given by

$$\hat{w} = \beta(1-\beta)(1-\lambda) V^U.$$

Now the new thing is that the optimal policy defines a *Markov process* for the state $x_t \in X$. Now let us simplify assuming the distribution of wages is a discrete distribution with J possible realizations $\{\omega_1, \omega_2, ..., \omega_J\}$ and probabilities $\{\pi_1, \pi_2, ..., \pi_J\}$ (the c.d.f. is now a step function). Suppose $\omega_{\hat{\jmath}-1} < \hat{w} < \omega_{\hat{\jmath}}$.

Now we have a *Markov chain* with transition probabilities given as follows:

$$\begin{aligned}
\Pr\left(x_{t+1} = \textit{unemployed} \mid x_t = \textit{unemployed}\right) &= \sum_{j=1}^{\hat{\jmath}-1} \pi_j \\
\Pr\left(x_{t+1} = \omega_j \mid x_t = \textit{unemployed}\right) &= 0 \text{ for } j = 1, ..., \hat{\jmath}-1 \\
\Pr\left(x_{t+1} = \omega_j \mid x_t = \textit{unemployed}\right) &= \pi_j \text{ for } j = \hat{\jmath}, ..., J \\
\Pr\left(x_{t+1} = \textit{unemployed} \mid x_t = \omega_j\right) &= \lambda \text{ for all } j \\
\Pr\left(x_{t+1} = \omega_j \mid x_t = \omega_j\right) &= 1-\lambda \text{ for all } j \\
\Pr\left(x_{t+1} = \omega_{j'} \mid x_t = \omega_j\right) &= 0 \text{ for all } j' \neq j \text{ and all } j
\end{aligned}$$

We can then address questions like: suppose you have a large population of agents (with independent wage draws and separation shocks) and you start from some distribution on the state X, if the economy goes on for a while do you converge to some invariant distribution on X?

This is the analogous of the deterministic dynamics, but the notion of convergence is different. No steady state but invariant distribution.

Example: $\{\omega_1, \omega_2, \omega_3\}$ with $\hat{\jmath} = 2$, then $X = \{\textit{unemployed}, \omega_1, \omega_2, \omega_3\}$ and transition matrix:

$$M = \begin{bmatrix} \pi_1 & 0 & \pi_2 & \pi_3 \\ \lambda & 1-\lambda & 0 & 0 \\ \lambda & 0 & 1-\lambda & 0 \\ \lambda & 0 & 0 & 1-\lambda \end{bmatrix}.$$

Suppose you start from distribution $\left[\phi_{1,0}, \phi_{2,0}, \phi_{3,0}, \phi_{4,0}\right]'$. What happens to the distribution after t periods?

$$\begin{bmatrix} \phi_{1,t} \\ \phi_{2,t} \\ \phi_{3,t} \\ \phi_{4,t} \end{bmatrix} = M^t \begin{bmatrix} \phi_{1,0} \\ \phi_{2,0} \\ \phi_{3,0} \\ \phi_{4,0} \end{bmatrix}$$

Does it converge?