# 14.451 Lecture Notes 7

Guido Lorenzoni

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## 1 More on local stability

We now have the results needed to analyze local stability. From the 2nd order difference equation

$$z_{t+2} = \beta^{-1} F_{xy}^{-1} \left( F_{yy} + \beta F_{xx} \right) z_{t+1} + \beta^{-1} F_{xy}^{-1} F_{yx} z_t.$$

We get the matrix

$$M = \begin{bmatrix} \beta^{-1} F_{xy}^{-1} \left( F_{yy} + \beta F_{xx} \right) & \beta^{-1} F_{xy}^{-1} F_{yx} \\ I & 0 \end{bmatrix} = B^{-1} \Lambda B. \tag{1}$$

Suppose the eigenvalues on the diagonal of $\lambda$ are ordered from the smallest to the largest (in absolute value). So, given $z_0$, if we find a $z_1$ such that

$$B \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix}$$

for some $w_1 \in R^l$ and the first $l$ eigenvalues have absolute value smaller than 1, we have found a $z_1$ such that

$$M^j \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = B \Lambda^j \begin{bmatrix} w_1 \\ 0 \end{bmatrix} \to 0.$$

From this we construct a sequence $z_j = \begin{pmatrix} \mathbf{1} & 0 \end{pmatrix} M^j \begin{pmatrix} z_1 & z_0 \end{pmatrix}'$ that satisfy Euler and transversality and we are done.

From

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} w_1 \\ 0 \end{bmatrix},$$

we need to solve

$$B_{21} z_1 + B_{22} z_0 = 0.$$

To do so we show that $B_{21}$ is invertible. Proceeding by contradiction, suppose not. Then $B_{21}$ is singular and there is a $\tilde{z}_1 \neq 0$ such that

$$B_{21} \tilde{z}_1 = 0.$$

But then

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{w}_1 \\ 0 \end{bmatrix}$$

for some $\tilde{w}_1 \in R^l$. This means that $x_1 = x^* + \tilde{z}_1 \neq x^*$ is optimal from the initial condition $x_0 = x^* + 0 = x^*$. This contradicts the fact that $x^*$ is a steady state and $g(x^*) = x^*$ (the policy is unique by strict concavity).

We have all the crucial steps for the following theorem. But first let us remember our assumptions.

**Assumption 1.** *The payoff function $F(x, y)$ is quadratic and strictly concave. The feasible set is a convex set $X \subset R^l_+$, the constraint correspondence $\Gamma(x)$ is continuous.*

**Assumption 2**. *The matrices $F_{xy}$ and $F_{yx} + F_{yy} + \beta F_{xx} + \beta F_{xy}$ are nonsingular.*

**Assumption 3.** *The steady state*

$$x^* = (F_{yx} + F_{yy} + \beta F_{xx} + \beta F_{xy})^{-1} (F_y(0, 0) + \beta F_x(0, 0))$$

*is interior, $x^* \in int\Gamma(x^*)$, and satisfies $F_x > 0$.*

**Assumption 4.** *The matrix $M$ defined in (1) has $l$ eigenvalues smaller than* 1 *in absolute value.*

**Theorem 1** *Under assumptions 1 to 4, there is a neighborhood $\mathcal{I}$ of $x^*$ where the policy is given by*

$$g(x) = x^* - B_{21}^{-1} B_{22} (x - x^*).$$

*The optimal sequence converges to the steady state $x^*$ for any initial condition $x_0 \in \mathcal{I}$.*

**Proof.** All the steps before tell us that given any initial condition $x_0 \in R^l$ there is a sequence $\{x_t^*\}_{t=0}^{\infty}$ that satisfies the Euler equation at each $t$ and converges to $x^*$. By choosing $x_0$ sufficiently close to $x^*$ we can ensure that $x_t^* \in int\Gamma\left(x_{t-1}^*\right)$ and $F_x\left(x_t^*, x_{t+1}^*\right) > 0$ for all $t$ (make sure you know how to make this step more formal).

To complete the proof we use the argument in the last set of lecture notes showing that the Euler equation and the transversality condition are sufficient for an optimum. The Euler equation is satisfied by construction. The transversality condition holds because

$$\lim_{t\to\infty} \beta^t F_x\left(x_t^*, x_{t+1}^*\right) x_t = \lim_{t\to\infty} \beta^t F_x(x^*, x^*) x^* = 0.$$

Notice one small wrinkle: in last set of notes we assumed $F_x \geq 0$ everywhere, but fortunately the argument goes through if we only have $F_x\left(x_t^*, x_{t+1}^*\right) \geq 0$ along our candidate sequence $\{x_t^*\}_{t=0}^{\infty}$. (Notice that a quadratic objective rules out $F_x \geq 0$ everywhere, so we can't hope to use that argument). So the Euler equation and transversality are sufficient to show that $\{x_t^*\}_{t=0}^{\infty}$ is optimal. ■

Notice that if $X = R^l$ and $\Gamma(x) = R^l$ for all $x$, it is possible to prove that the preceding characterization of a quadratic problem holds globally, not only locally (proving it is a bit more involved that what it seems from reading SLP, because the transversality condition we are using here does not work when $X = R^l$).

### 1.1 The general nonlinear case

It was ok to restrict attention to local arguments in the quadratic case because we want to use the quadratic case as a step towards local characterization of steady states in the general case where the objective is possibly non-quadratic and so the policy is possibly non-linear. We now make the following assumptions, that generalize our previous set of assumptions.

**Assumption 1.** *The payoff function $F(x,y)$ is twice continuously differentiable and strictly concave. The feasible set is $X \subset R^l_+$, the constraint correspondence $\Gamma(x)$ is continuous.*

**Assumption 2**. *There is an $x^* \in X$ that satisfies $F_x(x^*, x^*) > 0$, $x^* \in int\Gamma(x^*)$ and*

$$F_y(x^*, x^*) + \beta F_y(x^*, x^*) = 0.$$

**Assumption 3.** *The matrices $F_{xy}$ and $F_{yx} + F_{yy} + \beta F_{xx} + \beta F_{xy}$, evaluated at $(x^*, x^*)$, are non-singular at the steady state.*

**Assumption 4.** *The matrix $M$, evaluated at $(x^*, x^*)$, has $l$ eigenvalues smaller than $1$ in absolute value.*

**Theorem 2** *Under assumptions 1 to 4, there is a neighborhood $\mathcal{I}$ of $x^*$ such that the optimal sequence converges to the steady state $x^*$ for any initial condition $x_0 \in \mathcal{I}$.*

**Proof (sketch).** The idea here is to use the implicit function theorem to find a function $h(x,y)$ which solves the Euler equation

$$F_y(x,y) + \beta F_y(y, h(x,y)) = 0$$

for all $(x,y)$ in a neighborhood of $(x^*, x^*)$. The difference equation then is

$$\begin{bmatrix} x_{t+2} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} h(x_t, x_{t+1}) \\ x_{t+1} \end{bmatrix}$$

and the Jacobian of the map on the right is

$$\begin{bmatrix} h_1 & h_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} \beta^{-1} F_{xy}^{-1} (F_{yy} + \beta F_{xx}) & \beta^{-1} F_{xy}^{-1} F_{yx} \\ I & 0 \end{bmatrix}.$$

Then using general local characterization of non-linear difference equations (Thm 6.6) we can find a neighborhood $U$ of $(x^*, x^*)$ and a continuously differentiable function $\phi(x_t, x_{t+1})$ such that if $\phi(x_0, x_1) = 0$ and $x_{t+2} = h(x_t, x_{t+1})$ for all $t \geq 1$ then $\lim_{t\to\infty} x_t = x^*$. With a step analogous to the linear case we can show that for all $x_0$ there is a $x_1$ that satisfies $\phi(x_0, x_1) = 0$ and choose the neighborhood small enough that all $x_t$ are in the interior of the constraint set and have $F_x > 0$. ■