14.451 Exam

October 26, 2009

1 Short problems

Question 1

Suppose that the Markov chain z_t can assume only two values $\{z^h, z^l\}$ and has the following transition matrix

$$\Pi = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

(so that p is the probability of remaining in state z^s and $1-p$ is the probability of switching to the other state). Suppose that the state x_t can also assume only two values $\{x^h, x^l\}$ and that the optimal policy (of some dynamic optimization is given by

$$x_{t+1} = g(x_t, z_t) = \begin{cases} x^h & \text{if } (x_t, z_t) = (x^h, z^h) \text{ or } (x_t, z_t) = (x^h, z^l) \\ x^l & \text{if } (x_t, z_t) = (x^l, z^h) \text{ or } (x_t, z_t) = (x^l, z^l) \end{cases}.$$

1. Write down the transition matrix of the Markov chain (x_t, z_t).
2. How many ergodic states does the Markov chain (x_t, z_t) have? What can you say about its invariant distribution? (does it exist? is it unique?)

Question 2

Suppose you find a sequence $\{x_t^*\}$ which satisfies

$$V^*(x_{t+1}^*) = F(x_t^*, x_{t+1}^*) + \beta V^*(x_t^*)$$

and

$$\begin{aligned} x_{t+1}^* &\in \Gamma(x_t^*), \ \forall t \\ x_0^* &= x_0 \end{aligned}$$

where $V^*(x)$ is the *value function* of the problem. Is $\{x_t^*\}$ an optimal plan?

Question 3

Suppose that you have an infinite horizon, deterministic problem in discrete time with a *quadratic* objective function and have found a steady state $\bar{x}$. Let $\{z_t\}$ be the sequence defined as

$$z_t = x_t - \bar{x} \text{ for all } t.$$

From the Euler equation you have obtained the following first order difference equation

$$\begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} = B^{-1} \begin{bmatrix} (1/2)^t & 0 \\ 0 & (1.9)^t \end{bmatrix} B \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}.$$

Where B is a 2×2 matrix. Suppose that the vector $\begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$ is such that $\{z_t\}$ converges to 0, can you say anything about the vector $B \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$? What can you say about the optimal policy $g(x_0)$?

Question 4

Write down the Euler equation and the transversality condition for a generic discrete time deterministic problem. Suppose you find a sequence $\{x_t\}$ which solves the Euler equation and the transversality condition. When can you be sure that this sequence is an optimum for the sequential problem? (Remember to state all the assumptions).

Question 5

Consider the static problem of a firm that can produce 10 different goods using labor and capital. The technology to produce good i is described by the production function

$$y_i = f_i(k_i, l_i),$$

where k_i is units of capital, l_i is units of labor, and $f_i : R_+^2 \to R_+$ is a concave, differentiable function, strictly increasing in both arguments, which satisfies $f_i(k_i, l_i) = 0$ if either k_i or l_i is zero. The price of good i is taken as given by the firm and is equal to $p_i > 0$. Let $R(K, L)$ be the function that gives us the *optimal* revenue of the firm when K units of capital and L units of labor are available (the firm maximizes $\sum p_i y_i$ subject to the constraints $\sum l_i \leq L$ and $\sum k_i \leq K$).

1. Prove that the function R is concave.
2. Prove that R is differentiable if $K > 0$ and $L > 0$.

2 The Burglar Problem

Consider the following problem, set in discrete time. There is a burglar who in each period has to decide whether to do one more robbery or to retire. If he chooses to do the robbery, with probability α he is not caught and obtains a payoff of $r > 0$ to add to his wealth. With probability $1 - \alpha$ he is caught, loses all his accumulated wealth and goes to jail. The utility from going to jail is 0. The burglar wealth is all in cash, so it earns no interest. The burglar does not consume before retirement and when he retires he consumes all the wealth accumulated. He has linear utility and discounts future consumption at the rate $\beta < 1$. At date 0 he begins with an accumulated wealth of $w_0 \geq 0$.

1. To warm up, suppose first that the burglar is forced to retire after one period (at $t = 1$). What is the optimal policy in period 0? How does it depend on the initial wealth w_0?

2. Now go back to the original problem. Set up the problem in recursive form, choosing your state variables. Write the Bellman equation for this problem.

Now you want to prove that the one-stage policy found in part 1 is also optimal for the original problem.

3. Argue that the value function *after retirement* is $V^a(w) = w$. Conjecture that the value function *before retirement* satisfies the following property: $V^b(w) = w$ for $w \geq w^*$ and $V^b(w) > w$ for $w < w^*$ for *the same* cutoff w^* you obtained in part 1. Show that the policy in part 1 is optimal if V^b has this form.

4. Prove that the functional equation defined in part 2 has a solution of the form conjectured in part 3 (Hint: you need to use an inductive argument to prove that V^b has the desired form. Then restrict attention to the range $[0, w^*]$ and find $V^b(w)$ in that range with a contraction mapping argument.)

5. Argue that the solution to the functional equation found in 4 corresponds to the value function of the original problem.

6. Prove that there is a finite time T in which for sure the burglar is either retired or in jail.

3 Reaching the center in continuous time

Consider the following problem, set in continuous time. An agent is located at some point $x(0)$ on the segment $[-1, 1]$. The agent wants to reach point 0 but traveling is subject to convex costs. Namely, traveling at speed y costs y^2.

Moreover, each period the agent pays a cost x^2 for being at a distance x from point 0. So the agent objective is to maximize

$$\int_0^\infty e^{-\rho t}\left(-x\left(t\right)^2 - y\left(t\right)^2\right) dt$$

subject to

$$\dot{x}\left(t\right) = y\left(t\right)$$

$x\left(t\right) \in [-1, 1]$ and $x\left(0\right)$ given.

1. Setup the Hamiltonian and derive the optimality conditions (ignore the constraint $x\left(t\right) \in [-1, 1]$, you'll check it at the very end).

2. Prove that the maximized Hamiltonian is a strictly concave function of the state variable $x\left(t\right)$ (Arrow condition).

3. Conjecture that the optimal $y\left(t\right)$ is a linear function of $x\left(t\right)$, namely, that $y\left(t\right) = \alpha x\left(t\right)$ for some $\alpha < 0$. Derive the value of α that ensure that this policy satisfies the optimality conditions in part 1 (proving that this α exists is enough).

4. Using the result in part 2 and the appropriate transversality condition argue that you have found the unique optimal path.

14.451 Dynamic Optimization Methods with Applications
Fall 2009