14.30 Introduction to Statistical Methods in Economics
Spring 2009

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Appendix to Lecture Notes 10

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1 Example of Transformation Formula of Integration Limits

$$f_{xy} = \begin{cases} 4xy & \text{if } 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}$$

What is the p.d.f. of $Z = X/Y$?

1.1 Approach 1: '2-step' method, too complicated

- find (x, y) such that $x/y \leq 2$.
- integrate $f_{xy}(x, y)$ over those (x, y)'s to obtain c.d.f. $F_z(z)$
- differentiate $F_z(z)$ to obtain p.d.f. $f_z(z)$

$\longrightarrow$ We won't do this, we have an easier approach.

1.2 Approach 2: change-of-variable formula

- problem: $z = u_1(x, y) = x/y$ one-dimensional, $u(\cdot)$ can't be one-to-one.
- fix: introduce additional variable $w = u_2(x, y) = XY \longrightarrow$ can invert $\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix}$

$S_1(w, z) = \sqrt{wz} = \sqrt{xy \cdot \frac{x}{y}} = \sqrt{x^2} = X$

$S_2(w, z) = \sqrt{\frac{w}{z}} = \sqrt{\frac{xy}{x/y}} = \sqrt{y^2} = Y$ (Note that x,y are positive with probability 1.)

$\Rightarrow$ inverse function is $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} S_1(w,z) \\ S_2(w,z) \end{bmatrix} = \begin{bmatrix} \sqrt{WZ} \\ \sqrt{W/Z} \end{bmatrix}$.

$\Rightarrow$ Jacobian is $J = \begin{bmatrix} \frac{\partial S_1}{\partial W} & \frac{\partial S_1}{\partial Z} \\ \frac{\partial S_2}{\partial W} & \frac{\partial S_2}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{Z}{2\sqrt{WZ}} & \frac{W}{2\sqrt{WZ}} \\ \frac{1/Z}{2\sqrt{W/Z}} & -\frac{W/Z^2}{2\sqrt{W/Z}} \end{bmatrix}$.

$\Rightarrow$ $\det(J) = -\frac{ZW/Z^2}{4W} - \frac{W/Z}{4W} = -\frac{1}{2Z}$.

- Use formula to get joint p.d.f. of (W, Z).

$$
\begin{aligned}
f_{wz}(w,z) &= f_{xy}(s_1(w,z), s_2(w,z))\,|\det(J)| \\
&= \begin{cases} 4s_1(w,z)s_2(w,z)\cdot\left|-\frac{1}{2z}\right| & \text{if } 0 < s_1(w,z), s_2(w,z) < 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{4W}{2Z} = 2\frac{W}{Z} & \text{if } w,z > 0 \text{ and } \textit{both } (*) \begin{cases} W < Z \\ W < 1/Z \end{cases} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Condition (*) comes from

$$1 > s_1(w,z) = \sqrt{wz} \Rightarrow w < 1/z$$

and

$$1 > s_2 = \sqrt{w/z} \Rightarrow w < z$$

- How do we obtain $f_z(z) = \int\limits_{-\infty}^{\infty} f_{wz}(w,z)dw$?

  - $f_{wz}(w,z)$ zero for $W \leq 0$.
  - $F_{wz}(w,z)$ zero for $W > min(Z, 1/Z)$
  - therefore,

$$
\begin{aligned}
f_z(z) &= \int\limits_0^{\max(0,\min(z,1/z))} 2\frac{W}{Z}dw = \left|\frac{W^2}{Z}\right|_0^{\max(0,\min(z,1/z))} \\
&= \begin{cases} z & \text{if } 0 < z < 1/z & \Leftrightarrow & 0 \leq z < 1 \\ 1/z^3 & \text{if } 0 < 1/z < z & \Leftrightarrow & 1 \leq z \\ 0 & \text{if } z < 0 \end{cases}
\end{aligned}
$$