14.30 Introduction to Statistical Methods in Economics
Spring 2009

Problem Set #5

14.30 - Intro. to Statistical Methods in Economics

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Due: Tuesday, March 31, 2009

Question One

The convolution formula is a useful trick when we are interested in the sum or average of independent random variables. In the last problem set, we dealt with the random variable X, below.

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}.$$

Now, suppose that $X = X_1 = X_2 = \cdots = X_k$ are independent, identically distributed random variables.

1. Using the convolution formula, determine the PDF of $Y_2 = \frac{1}{2}(X_1 + X_2)$. *Hint: Define $Z_1 = X_1$ and $Z_2 = X_1 + X_2$ and then use the transformation method to get back to Y_2 from Z_2.*

2. Compute its expectation: $\mathbb{E}[Y_2]$.

3. Using the convolution formula, determine the PDF for $Y_3 = \frac{1}{3}(X_1 + X_2 + X_3)$. *Hint: Use the hint from part 1 to define $Z_3 = X_1 + X_2 + X_3$ and perform a convolution with X_3 and Z_2 to transform the problem into Z_2 and Z_3.*

4. Compute its expectation: $\mathbb{E}[Y_3]$.

5. Using the convolution formula, determine the PDF for $Y_k = \frac{1}{k}(X_1 + X_2 + \cdots + X_k) = \frac{1}{k}\sum_{i=1}^{k} X_k$. *Hint: Try to determine a pattern from part 1 and part 3 using the methods from their hints.*

6. Compute its expectation: $\mathbb{E}[Y_k]$.

7. What does this tell us about the mean of a sample of size k? Is this property specific to the exponential distribution? Explain.

Question Two

(Bain/Engelhardt, p. 228)

Suppose that $X_1, X_2, \ldots, X_k$ are independent random variables and let $Y_i = u_i(X_i)$ for $i = 1, 2, \ldots, k$. Show that $Y_1, Y_2, \ldots, Y_k$ are independent. Consider only the case where X_i is continuous and $y_i = u_i(x_i)$ is one-to-one. *Hint:* If $x_i = w_i(y_i)$ is the inverse transformation, then the Jacobian has the form

$$J = \prod_{i=1}^{k} \frac{d}{dy_i} w_i(y_i).$$

For extra credit, prove the *Hint* about the Jacobian.

Question Three

Moved to a later problem set.

Question Four

Order statistics are very useful tools for analyzing the properties of samples.

1. Write down the general formula of the pdf and cdf for the k^{th} order statistic of a sample of size n of a random variable X with CDF $F_X(x)$.

Question Five

(Bain/Engelhardt p. 229)

Let X_1 and X_2 be a random sample of size $n = 2$ from a continuous distribution with pdf of the form $f(x) = 2x$ if $0 < x < 1$ and zero otherwise.

1. Find the marginal pdfs of the smallest and largest order statistics, Y_1 and Y_2.
2. Compute their expectations, $\mathbb{E}[Y_1]$ and $\mathbb{E}[Y_2]$.
3. Find the joint pdf of Y_1 and Y_2.
4. Find the pdf of the sample range $R = Y_2 - Y_1$.
5. Compute the expectation of the sample range, $\mathbb{E}[R]$.

Question Six

(Bain/Engelhardt p. 229)

Consider a random sample of size n from a distribution with pdf $f(x) = \frac{1}{x^2}$ if $1 \leq x < \infty$; zero otherwise.

1. Give the joint pdf of the order statistics.

2. Give the pdf of the smallest order statistic, Y_1.

3. Compute its expectation, $\mathbb{E}[Y_1]$, or explain why it does not exist.

4. Give the pdf of the largest order statistic, Y_n.

5. Compute the expectation, $\mathbb{E}[X]$, of a single draw X from $f(x)$. Does the integral diverge? What does that say about the existence of $\mathbb{E}[Y_n]$? Explain.

6. Derive the pdf of the sample range, $R = Y_n - Y_1$, for $n = 2$. *Hint: Use partial fractions, searching on Yahoo for "QuickMath" will help you get the partial fractions via computer:*
   http://72.3.253.76:8080/webMathematica3/quickmath/page.jsp?s1=algebra&s2=partialfractions&s3=basic

7. Compute its expectation, $\mathbb{E}[R]$, or explain why it does not exist.

8. Give the pdf of the sample median, Y_r, assuming that n is odd so that $r = (n+1)/2 \in \mathbb{N}$. Express the pdf as a function of just r and y_r (eliminate all n's and k's).

9. Compute its expectation, $\mathbb{E}[Y_r]$, or explain why it does not exist.