Lecture Note 5 *

Random Variable/Vector Transformation

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13 Function of a Random Variable (Univariate Model)

13.1 Discrete Model

Let X be a discrete random variable with pmf $f_X(x)$. Define a new random variable Y as a function of X, $Y = r(X)$. The pmf of Y, $f_Y(y)$, is derived as follows:

$$f_Y(y) = P(Y = y) = P[r(X) = y] = \sum_{x:r(x)=y} f_X(x) \tag{31}$$

Example 13.1. Find $f_Y(y)$, where $Y = X^2$ and $P(X = x) = 0.2$ for $x = -2, -1, 0, 1, 2$, 0 if otherwise.

*Caution: These notes are not necessarily self-explanatory notes. They are to be used as a complement to (and not as a substitute for) the lectures.

13.2 Continuous Model

13.2.1 2-Step Method

Let X be a random variable with pdf $f_X(x)$. Define a new random variable Y as a function of X, $Y = r(X)$. The pdf of Y, $f_Y(y)$, is derived as follows:

$$
\begin{aligned}
1^{\text{st}} \text{ step}: \; F_Y(y) &= P(Y \leq y) = P[r(X) \leq y] = \int_{x:r(x)\leq y} f_X(x)dx \\
2^{\text{nd}} \text{ step}: \; f_Y(y) &= \frac{dF_Y(y)}{dy} \qquad \text{(at every point } F_Y(y) \text{ is differentiable).}
\end{aligned} \tag{32}
$$

Example 13.2. Find $f_Y(y)$, where $Y = X^2$ and $X \sim U[-1, 1]$.

13.2.2 1-Step Method

Let X be a random variable with pdf $f_X(x)$. Define the set $\mathcal{X}$ as all possible values of X such that $f_X(x) > 0$ [$\mathcal{X} = \{x : f_X(x) > 0\}$; for example: $a < X < b$].

Define a new random variable Y, such that $Y = r(X)$, where $r(\,)$ is a strictly monotone function (increasing or decreasing) and a differentiable (and thus continuous) function of X. Then, the pdf of Y, $f_Y(y)$, is derived as follows:

$$
f_Y(y) = \begin{cases} f_X\Big(r^{-1}(y)\Big) \left| \dfrac{\partial r^{-1}(y)}{\partial y} \right| , & \text{for } y \in \mathcal{Y} \subseteq R; \\ 0 \,, & \text{otherwise.} \end{cases} \tag{33}
$$

Where the set $\mathcal{Y}$ is defined as: $\mathcal{Y} = \{y : y = r(x) \text{ for all } x \in \mathcal{X}\}$. For example: $a < X < b \Longleftrightarrow \alpha < Y < \beta$.

- If $r(x)$ is not monotonic, find a partition of X such that each segment is monotonic. Then, apply the method to each segment and aggregate.

- Where does formula (33) come from?

Example 13.3. Find $f_Y(y)$, where $Y = 4X + 3$ and $f(x) = 7e^{-7x}$ if $0 < x < \infty$, 0 if otherwise.

Example 13.4. Do Example **13.2** using the 1-step method.

14 Function of a Random Vector (Multivariate Model)

14.1 Discrete Model

Let $\mathbf{X} \equiv (X_1, X_2, ..., X_n)$ be a random vector with joint pmf $f_{\mathbf{X}}(x_1, ..., x_n)$.

Define a new random vector $\mathbf{Y} \equiv (Y_1, Y_2, ..., Y_m)$ as a function of the random vector $\mathbf{X}$, such that $Y_i = r_i(X_1, X_2, ..., X_n)$ for $i = 1...m$. The joint pmf of $\mathbf{Y}$, $f_{\mathbf{Y}}(y_1, y_2, ..., y_m)$, is derived as follows:

$$f_{\mathbf{Y}}(y_1, y_2, ..., y_m) = \sum_{\substack{(x_1,...,x_n)\ :\ r_i(x_1,...,x_n)=y_i \\ \forall i=1..m}} f_{\mathbf{X}}(x_1, ..., x_n) \tag{34}$$

- This is a direct generalization of section 13.1, where (34) is the generalization of (31).

Example 14.1. (Convolution) Let (X, Y) be a random vector, such that X and Y are independent and discrete RVs with pmf $f_X(x)$ and $f_Y(y)$. Find $P(Z = z)$, where $Z = Y + X$.

14.2 Continuous Model

14.2.1 2-Step Method

Let $\mathbf{X} \equiv (X_1, X_2, ..., X_n)$ be a random vector with joint pdf $f_{\mathbf{X}}(x_1, ..., x_n)$.

Define a new random vector $\mathbf{Y} \equiv (Y_1, ..., Y_m)$ as a function of the random vector $\mathbf{X}$, such that $Y_i = r_i(X_1, X_2, ..., X_n)$ for $i = 1, ..., m$. The joint pdf of $\mathbf{Y}$, $f_{\mathbf{Y}}(y_1, ..., y_m)$, is derived as follows (for the case where $m = 1$):

$$\begin{aligned} 1^{\text{st}} \text{ step} : F_Y(y) &= P(Y \leq y) = P[r(X_1, ..., X_n) \leq y] = \int ... \int_{(\mathbf{x}):r(\mathbf{x})\leq y} f_{\mathbf{X}}(x_1, ..., x_n) dx_1 ... dx_n \\ 2^{\text{nd}} \text{ step} : f_Y(y) &= \frac{dF_Y(y)}{dy} \qquad \text{(at every point } F_Y(y) \text{ is differentiable.)} \end{aligned} \tag{35}$$

- This is a direct generalization of section 13.2.1, where (35) is the generalization of (32) (for the case where $m = 1$).

- The case where $m > 1$ is analogous (but more messier).

14.2.2 1-Step Method

Let $\mathbf{X} \equiv (X_1, X_2, ..., X_n)$ be a random vector with joint pdf $f_{\mathbf{X}}(x_1, ..., x_n)$.

Define a new random vector $\mathbf{Y} \equiv (Y_1, ..., Y_n)$ as a function of the random vector $\mathbf{X}$, such that $Y_i = r_i(X_1, X_2, ..., X_n)$ for $i = 1, ..., n$, where condition (37) holds. The joint pdf of $\mathbf{Y}$, $f_{\mathbf{Y}}(y_1, ..., y_n)$, is derived as follows:

$$f_{\mathbf{Y}}(y_1, y_2, ..., y_n) = \begin{cases} f_{\mathbf{X}}\big(s_1(), s_2(), ..., s_n()\big)\ |J| \ , & \text{for } (y_1, y_2, ..., y_n) \in \mathcal{Y} \subseteq R^n; \\ 0 \ , & \text{otherwise.} \end{cases} \tag{36}$$

where

$$\begin{array}{rcl} Y_1 = r_1(X_1, ...X_n) & & X_1 = s_1(Y_1, ...Yn) \\ Y_2 = r_2(X_1, ...X_n) & \textbf{unique} & X_2 = s_2(Y_1, ...Yn) \\ \vdots & \textbf{transformation} & \vdots \\ Y_n = r_n(X_1, ...X_n) & \longrightarrow & X_n = s_n(Y_1, ...Y_n); \end{array} \tag{37}$$

and

$$J = det \begin{bmatrix} \frac{\partial s_1}{\partial y_1} & \cdots & \frac{\partial s_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_n}{\partial y_1} & \cdots & \frac{\partial s_n}{\partial y_n} \end{bmatrix} \qquad (Jacobian); \tag{38}$$

and

$$\begin{array}{ll} \mathcal{X} & \text{is the support of } X_1, ...X_n : \mathcal{X} = \{\mathbf{x} : f_{\mathbf{X}}(\mathbf{x}) > 0\}. \\ \mathcal{Y} & \text{is the induced support of } Y_1, ...Y_n : \mathcal{Y} = \{\mathbf{y} : \mathbf{y} = r(\mathbf{x}) \text{ for all } \mathbf{x} \in \mathcal{X}\}. \\ & (x_1, ...x_n) \in \mathcal{X} \Longleftrightarrow (y_1, ...y_n) \in \mathcal{Y}. \end{array} \tag{39}$$

- Note that for this method to work, m has to be equal to n ($n = m$).

- If condition (37) does not hold, find a partition such that it holds in each segment. Then, apply the method to each segment and aggregate.

- This is a direct generalization of 13.2.2, where (36) is the generalization of (33).

- Reminder: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $det(A) = |A| = ad - cb$.

Example 14.2. Let (X_1, X_2) be a random vector, such that X_1 and X_2 are continuous RVs with joint pdf $f(x_1, x_2) = e^{-x_1-x_2}$ if $0 \leq x_i$, and 0 if otherwise. Using the 1-step method find $f_Y(y)$, where $Y = X_1 + X_2$.

Example 14.3. Let $(X_1, X_2, ..., X_n)$ be a continuous random vector containing n independent and identically distributed random variables, [1] where $X_i \sim U[0, 1]$. Compute the pdf of the following two transformations of the random vector $\mathbf{X}$: i) $Y_{max} = max\{X_1, X_2, ..., X_n\}$ and ii) $Y_{min} = min\{X_1, X_2, ..., X_n\}$.

[1] *iid* for short or also called "random sample." More on this in Lecture Note 7.