14.30 Exam 2 Solutions
Fall 2004
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1 A:

a)

$E(aX + bY + c) = aE(X) + bE(Y) + c$

$\iint (aX + bY + c)f_{X,Y}(x, y)dxdy = \int\int aXf_{X,Y}(x, y) + \int\int bYf_{X,Y}(x, y) + \int\int cf_{X,Y}(x, y)$

$= \int aX \left[\int f_{X,Y}(x, y)dy\right] dx + \int bY \left[\int f_{X,Y}(x, y)dx\right] dy + c = \int aXf_X(x) + \int bYf_Y(y)dy + c$

$= aE(X) + bE(Y) + c$

$Q.E.D.$

b)

$E[E(aY \mid X)] = aE(Y)$

$\int E(aY \mid X)f_X(x)dx = \int \left[\int ayf(y \mid x)dy\right] f_X(x)dx$

$a\int\int Yf(y \mid x)f_X(x)dydx = a\int Y \left[\int f(y \mid x)f_X(x)dx\right] dy$

$a\int Y \left[\int f_{X,Y}(x, y)dx\right] dy = a\int Yf_Y(y)dy$

$= aE(Y)$

$Q.E.D.$

c)

First lets show: $Cov(X + Y, X - Y) = VarX - VarY$

$Cov(X + Y, X - Y) = E[(X + Y)(X - Y)] - E(X + Y)E(X - Y)$

$EX^2 - EY^2 - E^2(X) + E^2(Y) = EX^2 - E^2(X) - [EY^2 - E^2(Y)]$

$= VarX - VarY$

now, $Corr(X + Y, X - Y) = \frac{Cov(X+Y,X-Y)}{\sqrt{Var(X+Y).Var(X-Y)}}$

we solved the numerator, now lets look at the denominator:

$\sqrt{Var(X + Y).Var(X - Y)} = \sqrt{[VarX + VarY + 2Cov(X, Y)] . [VarX + VarY - 2Cov(X, Y)]}$

$\sqrt{Var^2X + Var^2Y + 2VarX.VarY} = \sqrt{(VarX + VarY)^2}$

$= VarX + VarY$

$Q.E.D.$

1 B:

$VarX = EX^2 - E^2X$

$EX = 0.0.4 + 1.0.6 = 0.6$

$EX^2 = 0^2.0.4 + 1^2.0.6 = 0.6$

$VarX = 0.6 - 0.36 = 0.24$

Similarly, $EY = 0.6$
$E(Z) = E(XY) = \sum xyf(x,y) = 0.25$
$Cov(X,Y) = E(XY) - EXEY = 0.25 - 0.36 = -0.11$

2 a:
$X \sim U(1,3)$
$f_X(x) = \frac{1}{3-1} = \frac{1}{2}$
$Y = -\alpha \ln(3X)$

Now in order to apply the 1-step method, we need to first confirm that this function is monotonic:
$\frac{dY}{dX} = -\frac{\alpha}{3X} < 0 \quad$ for $1 < X < 3$
which is monotone decreasing function in the range of X.
Range of y:
$1 < X < 3$
$-\alpha \ln 9 < y < -\alpha \ln 3$
where:
$Y = -\alpha \ln(3X)$
$X = \frac{1}{3}e^{-\frac{Y}{\alpha}}$
now 2-step method would be:
$F_Y(y) = 1 - F_X(x) = 1 - \int_1^{\frac{1}{3}e^{-\frac{Y}{\alpha}}} \frac{1}{2}dx$
$= 1 - \frac{1}{6}e^{-\frac{Y}{\alpha}} + \frac{1}{2} = \frac{3}{2} - \frac{1}{6}e^{-\frac{Y}{\alpha}}$
$\frac{dF_Y(y)}{dy} = f_Y(y) = \left\{ \begin{array}{ll} \frac{1}{6\alpha}e^{-\frac{Y}{\alpha}} & \text{for } -\alpha\ln 9 < y < -\alpha \ln 3 \\ 0 & \text{otherwise} \end{array} \right\}$
and 1-step method would be:
$f_X\left(r^{-1}(y)\right) = \frac{1}{2}$
$\left|\frac{dr^{-1}(y)}{dy}\right| = \frac{1}{3\alpha}e^{-\frac{Y}{\alpha}}$
$f_Y(y) = f_X\left(r^{-1}(y)\right).\left|\frac{dr^{-1}(y)}{dy}\right| = \left\{ \begin{array}{ll} \frac{1}{6\alpha}e^{-\frac{Y}{\alpha}} & \text{for } -\alpha\ln 9 < y < -\alpha \ln 3 \\ 0 & \text{otherwise} \end{array} \right\}$

2 b:
$f(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}$ for iid X_1 and X_2
$y = \max\{aX_1, X_2 + c\}$
$P(Y \le y) = P(aX_1 \le y).P(X_2 + c \le y)$
$P(X_1 \le \frac{y}{a}).P(X_2 \le y - c)$
$= \left[\int_0^{\frac{y}{a}} \frac{1}{\beta}e^{-\frac{x}{\beta}}dx\right].\left[\int_0^{y-c} \frac{1}{\beta}e^{-\frac{x}{\beta}}dx\right]$
$(1 - e^{-\frac{y}{a\beta}}).(1 - e^{-\frac{y-c}{\beta}}) = 1 - e^{-\frac{y-c}{\beta}} - e^{-\frac{y}{a\beta}} + e^{-\frac{y(a+1)-ac}{a\beta}} = F_Y(y)$
$f_Y(y) = \frac{dF_Y(y)}{dy} = \left\{ \begin{array}{ll} \frac{e^{-\frac{y-c}{\beta}}}{\beta} + \frac{e^{-\frac{y}{a\beta}}}{a\beta} - \frac{(a+1)e^{-\frac{y(a+1)-ac}{a\beta}}}{a\beta} & \text{if } y\epsilon(c,\infty) \\ 0 & \text{otherwise} \end{array} \right\}$

3 a:

$X_{Boston} \sim N(65, 9)$
$X_{Santiago} \sim N(60, 4)$
$P(X_B < 66.5) = P(\mu + \sigma Z < 66.5)$
where $Z \sim N(0, 1)$
$= P(Z < \frac{66.5-65}{3})$
$= P(Z < 0.5) = 0.6915$

So Probability randomly chosen woman is taller than Alice $= (1 - 0.6915) = 0.3085$

Number of women in Boston taller than Alice: $0.3085 * 2,000,000 = 617,000$

3 b:

Distribution of "sum of independent normally distributed variables" is also normal:

so, $\left(X_B^1 + X_B^2 + X_S^1 + X_S^2 + X_S^3\right) \sim N(\mu_{sum}, \sigma_{sum}^2) \sim N(2\mu_B + 3\mu_S, 2\sigma_B^2 + 3\sigma_B^2)$
$= N(310, 30)$

3 c:

we want: $P(-1 < \overline{X} - \mu < 1)$
where $\overline{X}$ is the average height and μ is the population mean
$= P(\frac{-1}{\frac{2}{\sqrt{n}}} < \frac{\overline{X}-\mu}{\frac{2}{\sqrt{n}}} < \frac{1}{\frac{2}{\sqrt{n}}})$
$= P(\frac{-\sqrt{n}}{2} < Z < \frac{\sqrt{n}}{2})$
$= 2P(Z < \frac{\sqrt{n}}{2}) - 1$
We want at least 95% probability:
$= 2P(Z < \frac{\sqrt{n}}{2}) - 1 \geq 0.95$
$P(Z < \frac{\sqrt{n}}{2}) \geq 0.975$
$\frac{\sqrt{n}}{2} \geq 1.96$
$n \geq 16$

4 a:

$X \sim N(50, 100)$
$P(40 < X < 60)$
$= P(\frac{40-50}{10} < Z < \frac{60-50}{10}) = P(-1 < Z < 1)$
$= 2P(Z < 1) - 1$
$= 2(0.8413) - 1 = 0.6826$

4 b:

New Technology results in: $X \sim N(50, \sigma^2)$, where is σ^2 unknown
We know: $P(40 < X < 60) = 0.95$
$P(\frac{40-50}{\sigma} < \frac{X-50}{\sigma} < \frac{60-50}{\sigma}) = 0.95$
$P(\frac{-10}{\sigma} < Z < \frac{10}{\sigma}) = 0.95$
$2P(Z < \frac{10}{\sigma}) - 1 = 0.95$
$P(Z < \frac{10}{\sigma}) = 0.975$
$\frac{10}{\sigma} = 1.96$
$\sigma = 5.102$
so change in standard deviation is: $10 - 5.102 = 4.898$

4 c:

Notice that this question points you towards another familiar distribution, that is the binomial distribution. We are given the number of observations,and we calculated the probability of success in part a) of this question. So we need to plug in:

$n = 555$; $p = 0.6826$; and $E(Y) = np; Var(Y) = np(1-p)$; so: $E(Y) = 378.843; Var(Y) = 120.245$

where Y is represented by the binomial distribution.

Define event M=255 cathodes out of 555 satisfy customer's specifications, then:

$$P(M = m \mid n, p) = \begin{pmatrix} n \\ m \end{pmatrix} p^m(1-p)^{n-m} = \begin{pmatrix} 555 \\ 255 \end{pmatrix} 0.6826^{255}(1-0.6826)^{555-255}$$

4 d:

A copper cathode of law L has a price of $\frac{3}{2}L^2$ cents.
Expected price: $\int \frac{3}{2}L^2 f_X(x)dx$
$= \frac{3}{2}\int L^2 f_X(x)dx = \frac{3}{2}E(X^2) = \frac{3}{2}\left[Var(X) + E^2(X)\right]$
$= \frac{3}{2}[100 + 2500] = 3900 = \$39.00$