14.30 EXAM 2 - SUGGESTED ANSWERS

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Question 1

A.

(i) $False$. The result $E(g(X)) = g(E(X))$ only holds for linear functions of X, because integration is only distributive over linear functions.

(ii) $False$. $\int_{-\infty}^{\infty} h(x) f_X(x)\, dx = E(H) = \int_{-\infty}^{\infty} h f_H(h)\, dh$.

(iii) $False$. The bivariate normal is a special case for which $\rho(X,Y) = 0$ implies independence, but in general, $\rho(X,Y) = 0$ does not imply the independence of X and Y.

(iv) $False$. If you can partition the range of X into regions over which $h(X)$ is monotone, you can apply the 1-step method even if $h(X)$ is not monotone over the entire range of X.

B.

(i) Applying our formula for expected values, we find $E(Z_1) = \sum_{f(z_1,z_2)>0} z_1 f(z_1, z_2) = 0(0.1) + 0(0.4) + 1(0.3) + 1(0.2) = 0.5$. Then, to find the conditional expectation, we will need the conditional distribution $f(z_1|Z_2 = 0) = \begin{matrix} \frac{0.1}{0.1+0.3} = 0.25 \text{ if } z_1 = 0 \\ \frac{0.3}{0.1+0.3} = 0.75 \text{ if } z_1 = 1 \\ 0 \quad \text{elsewhere} \end{matrix}$. Then we can calculate $E(Z_1|Z_2 = 0) = \sum_{f(z_1|Z_2=0)>0} z_1 f(z_1|Z_2 = 0) = 0(0.25) + 1(0.75) = 0.75$. For Y, we use the formula for the expected value of a function.

$E(Y = Z_1 + Z_2) = \sum_{f(z_1,z_2)>0} y f(z_1, z_2) = \sum_{f(z_1,z_2)>0} (z_1 + z_2) f(z_1, z_2) = 0(0.1) + 1(0.4) + 1(0.3) + 2(0.2) = 1.1$

(ii) Here we will apply our normal variance formula: $V(Z_1|Z_2 = 0) = E\left(Z_1^2|Z_2 = 0\right) - E(Z_1|Z_2 = 0)^2$. We first calculate $E\left(Z_1^2|Z_2 = 0\right) = 0(0.25) + 1(0.75) = 0.75$, and then we have $V(Z_1|Z_2 = 0) = 0.75 - 0.75^2 = \frac{3}{4} - \frac{9}{16} = \frac{3}{16}$. Our covariance formula is $Cov(Z_1, Z_2) = E(Z_1 Z_2) - E(Z_1) E(Z_2)$. We find $E(Z_1 Z_2) = \sum_{f(z_1,z_2)>0} (z_1 z_2) f(z_1, z_2) = 0(0.1) + 0(0.4) + 0(0.3) + 1(0.2) = 0.2$. Then, $E(Z_2) = \sum_{f(z_1,z_2)>0} z_2 f(z_1, z_2) = 0(0.1) + 1(0.4) +$

$0\,(0.3) + 1\,(0.2) = 0.6$. We plug all of this into the covariance formula: $Cov\,(Z_1, Z_2) = 0.2 - (0.5)\,(0.6) = -0.1$.

Question 2

A.

(*i*) We want to prove that if X and Y are independent, $E\,(aX + bY + c) = aE\,(X) + bE\,(Y) + c$. We start by using the definition of the expected value of a function, and then use the properties of integrals to proceed.

$$
\begin{aligned}
E\,(aX + bY + c) &= \int_X \int_Y (ax + by + c)\, f_{X,Y}\,(x, y)\, dydx \\
&= \int_X \int_Y ax f_{X,Y}\,(x, y)\, dydx + \int_X \int_Y by f_{X,Y}\,(x, y)\, dydx \\
&\quad + \int_X \int_Y c f_{X,Y}\,(x, y)\, dydx \\
&= a \int_X \int_Y x f_X\,(x)\, f_Y\,(y|x)\, dydx + b \int_Y \int_X y f_X\,(x|y)\, f_Y\,(y)\, dydx \\
&\quad + c \int_X \int_Y f_{X,Y}\,(x, y)\, dydx \\
&= a \int_X x f_X\,(x) \int_Y f_Y\,(y|x)\, dydx + b \int_Y y f_Y\,(y) \int_X f_X\,(x|y)\, dxdy + c \\
&= a \int_X x f_X\,(x)\, dx + b \int_Y y f_Y\,(y)\, dy + c \\
&= aE\,(X) + bE\,(Y) + c
\end{aligned}
$$

Note that terms such as $\int_Y f_Y\,(y|x)\, dy$ disappear because the integral of a pdf over its entire support must be equal to one.

(ii) We start by using a variance theorem, and then apply the property of expectations that we proved in part (i):

$$\begin{aligned}
Var\left(aX+Y+c+Z+d\right) &= E\left(\left(aX+Y+c+Z+d\right)^2\right)-\left(E\left(aX+Y+c+Z+d\right)\right)^2 \\
&= E\begin{pmatrix} a^2X^2+2aXY+2a\left(c+d\right)X+2aXZ+Y^2 \\ +2\left(c+d\right)Y+2YZ+2\left(c+d\right)Z+\left(c+d\right)^2 \end{pmatrix} \\
&\quad -\left(aE\left(X\right)+E\left(Y\right)+E\left(Z\right)+c+d\right)^2 \\
&= a^2E\left(X^2\right)+2aE\left(XY\right)+2a\left(c+d\right)E\left(X\right)+2aE\left(XZ\right)+E\left(Y^2\right) \\
&\quad +2\left(c+d\right)E\left(Y\right)+2E\left(YZ\right)+2\left(c+d\right)E\left(Z\right)+\left(c+d\right)^2 \\
&\quad -\begin{pmatrix} a^2E\left(X\right)^2+2aE\left(X\right)E\left(Y\right)+2a\left(c+d\right)E\left(X\right) \\ +2aE\left(X\right)E\left(Z\right)+E\left(Y\right)^2+2\left(c+d\right)E\left(Y\right) \\ +2E\left(Y\right)E\left(Z\right)+2\left(c+d\right)E\left(Z\right)+\left(c+d\right)^2 \end{pmatrix} \\
&= a^2\left(E\left(X^2\right)-E\left(X\right)^2\right)+2a\left(E\left(XY\right)-E\left(X\right)E\left(Y\right)\right) \\
&\quad +2a\left(E\left(XZ\right)-E\left(X\right)E\left(Z\right)\right)+\left(E\left(Y^2\right)-E\left(Y\right)^2\right) \\
&\quad +2\left(E\left(YZ\right)-E\left(Y\right)E\left(Z\right)\right) \\
&= a^2Var(X)+2aCov(X,Y)+2aCov\left(X,Z\right)+Var(Y) \\
&\quad +Var\left(Z\right)+2Cov\left(Y,Z\right)
\end{aligned}$$

Alternatively, we could have arrived at this result by using some of the properties of variance presented in class:

$$\begin{aligned}
Var\left(aX+Y+c+Z+d\right) &= Var\left(aX+Y+Z\right)+Var\left(c+d\right) \\
&= Var\left(aX+Y+Z\right) \\
&= Var\left(aX+Y\right)+Var\left(Z\right)+2Cov\left(aX+Y,Z\right) \\
&= a^2Var(X)+2aCov(X,Y)+Var(Y) \\
&\quad +Var\left(Z\right)+2Cov\left(aX+Y,Z\right) \\
&= a^2Var(X)+2aCov(X,Y)+Var(Y) \\
&\quad +Var\left(Z\right)+2aCov\left(X,Z\right)+2Cov\left(Y,Z\right)
\end{aligned}$$

Where the last line holds because

$$\begin{aligned}
Cov\left(aX+Y,Z\right) &= E(aXZ+YZ)-E\left(aX+Y\right)E\left(Z\right) \\
&= a\left(E(XZ)-E\left(X\right)E\left(Z\right)\right)+E\left(YZ\right)-E\left(Y\right)E\left(Z\right) \\
&= aCov\left(X,Z\right)+Cov\left(Y,Z\right)
\end{aligned}$$

Then, because X, Y and Z are independent, we know that $Cov(X,Y) = Cov\left(X,Z\right)=Cov\left(Y,Z\right)=0$, so

$$Var\left(aX+Y+c+Z+d\right)=a^2Var(X)+Var(Y)+Var(Z)$$

(iii) We did not rely on the independence of X and Y at any point in the proof of part (i), so this result will still hold if X and Y are not independent. However, the result in part (ii) will not generally hold if X and Y are not independent, because the covariance term will not (in general) be equal to zero.

B.

(i) Because we know $E(Y|X)$, we can use the law of iterated expectations to find $E(XY)$:

$$\begin{aligned} E(XY) &= E_X(E_Y(XY|X)) \\ &= E_X\left(aX^2 + bX\right) \\ &= aE\left(X^2\right) + bE(X) \end{aligned}$$

where the second line makes use of the fact that we can treat X as a constant when taking the expectation conditional on X.

(ii) Because Z has a standard normal distribution, Z^2 has a χ^2 distribution with one degree of freedom. Also, since we are interested in the pdf of W given the value of V, we don't have to worry about the distribution of V, and can just treat it as a constant. We have the transformation

$$W = V^2 + Z^2$$

which yields the inverse transformation

$$Z^2 = W - V^2$$

Because our original transformation is monotone,we can use the one-step method to get the pdf of W.

$$\begin{aligned} f_W(w|v) &= \begin{matrix} f_{Z^2}\left(w - v^2\right)\left|\frac{d}{dw}\left(w - v^2\right)\right| & w \geq v^2 \\ 0 & \text{elsewhere} \end{matrix} \\ &= \begin{matrix} \frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{2}}\left(w - v^2\right)^{-\frac{1}{2}} e^{-\left(\frac{w - v^2}{2}\right)} & w \geq v^2 \\ 0 & \text{elsewhere} \end{matrix} \end{aligned}$$

Question 3

(i) Let D be a random variable that represents the deviation from the industry standard mineral law for a particular cathode of copper produced by Coldeco. Then $D \sim N(0, 225)$. We are interested in the probability that D is greater than or equal to -10. We set up the probability expression, convert it to the form necessary to use the standard normal distribution table, and look up the probability value.

$$\begin{aligned} \Pr(D \geq -10) &= \Pr(D \leq 10) \\ &= \Pr\left(\frac{D - 0}{15} \leq \frac{10 - 0}{15}\right) \\ &= \Pr(Z \leq 0.667) \\ &= 0.7486 \end{aligned}$$

(ii) Now we need to find the standard deviation such that the probability in part *(i)* would be equal to 0.9. In other words, we need to find σ such that $\Pr(D \geq -10) = \Pr\left(Z \leq \frac{10}{\sigma}\right) = 0.9$. From the table, we see that $\Pr(Z \leq 1.29) = 0.9$. So $\sigma = \frac{10}{1.29} = 7.75$. Thus the standard deviation of the production process needs to decrease from 15 percentage points to 7.75 percentage points, or by 7.25 percentage points.

(iii) The number of cathodes that fit the customer's specifications out of a number n that the customer purchases will be a binomial random variable; let's call it X. We want to find the smallest n such that $\Pr(X \geq 255) = 0.99$. Because $\Pr(X \geq 255) = \sum_{x=255}^{n} \Pr(X = x)$, and $\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, where p is the probability calculated in part *(i)*, we find n by finding the smallest integer that satisfies the following inequality:

$$\sum_{x=255}^{n} \binom{n}{x} 0.7486^x (0.2514)^{n-x} \geq 0.99$$

(iv) Knowing that the industry standard is priced at 600 dollars allows us to solve for the industry standard ($\overline{L}$):

$$\begin{aligned} \frac{3}{2}\overline{L}^2 &= 600 \\ \overline{L}^2 &= 400 \\ \overline{L} &= 20 \end{aligned}$$

We know that D, the percentage point deviation from the industry standard of the mineral law of a copper cathode produced by Coldeco, is distributed normally with mean 0 and standard deviation 15. Therefore, we can find the distribution of the mineral law of each copper cathode (L) by performing a transformation from D to L, and using the property that a linear transformation of a normally distributed random variable is also normally distributed.

$$\begin{aligned} D &= \left(\frac{L - \overline{L}}{\overline{L}}\right) 100 \sim N(0, 225) \\ \frac{\overline{L}}{100} D + \overline{L} &= L \sim N\left(\overline{L}, \frac{\overline{L}^2 225}{100^2}\right) \\ L &\sim N(20, 9) \end{aligned}$$

Then, because it can be difficult to solve the integral expressions for expected values of functions of normally distributed random variables, we will use a shortcut to find the expected value of the price. We know that

$$Var(L) = E\left(L^2\right) - E(L)^2$$

which implies

$$E\left(L^2\right) = Var(L) + E\left(L\right)^2$$

Then

$$\begin{aligned} E\left(P\right) &= E\left(\frac{3}{2}L^2\right) \\ &= \frac{3}{2}\left(Var(L) + E\left(L\right)^2\right) \\ &= \frac{3}{2}\left(9+400\right) \\ &= \frac{1227}{2} = 613.5 \end{aligned}$$

Question 4

(i) First, we need to find the expected value and variance of Y.

$$\begin{aligned} E\left(Y\right) &= E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left(X_i\right) \\ &= \mu_i = 1 \\ Var\left(Y\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(X_i\right) \\ &= \frac{\sigma_i^2}{n} = \frac{4}{n} \end{aligned}$$

Now, we are interested in finding n such that $\Pr\left(\left|Y - E\left(Y\right)\right| \leq 1\right) \geq 0.99$. The Chebyshev inequality tells us that

$$\Pr\left(\left|Y - E\left(Y\right)\right| \geq 1\right) \leq \frac{Var\left(Y\right)}{1}$$

We can manipulate this expression to find the probability of interest.

$$\begin{aligned} \Pr\left(\left|Y - E\left(Y\right)\right| \geq 1\right) &\leq \frac{4}{n} \\ 1 - \Pr\left(\left|Y - E\left(Y\right)\right| \leq 1\right) &\leq \frac{4}{n} \\ \Pr\left(\left|Y - E\left(Y\right)\right| \leq 1\right) &\geq 1 - \frac{4}{n} \end{aligned}$$

So, we know that $\Pr\left(\left|Y - E\left(Y\right)\right| \leq 1\right) \geq 0.99$ for any n such that $1 - \frac{4}{n} \geq 0.99$, or $n \geq 400$.

(ii) If we know that X_i is normally distributed, we can find the exact probability that Y is within one unit of its mean for any value of n.

$$\begin{aligned}\Pr\left(\left|Y-1\right|\leq 1\right) &= 1-2\Pr\left(Y-1\geq 1\right)\\ &= 1-2\Pr\left(\frac{Y-1}{2/\sqrt{n}}\geq\frac{1}{2/\sqrt{n}}\right)\\ &= 1-2\Pr\left(Z\geq\frac{\sqrt{n}}{2}\right)\end{aligned}$$

We want to find all integers n that satisfy the following equivalent expressions:

$$\begin{aligned}1-2\Pr\left(Z\geq\frac{\sqrt{n}}{2}\right) &\geq 0.99\\ \Pr\left(Z\geq\frac{\sqrt{n}}{2}\right) &\leq 0.005\\ 1-\Pr\left(Z\leq\frac{\sqrt{n}}{2}\right) &\leq 0.005\\ \Pr\left(Z\leq\frac{\sqrt{n}}{2}\right) &\geq 0.995\end{aligned}$$

Using the table, we find that this is equivalent to $\frac{\sqrt{n}}{2}\geq 2.58$, or $n\geq 27$.