# Reputation

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# Game with Short-Run Players

$(N, A, u)$: two-player normal-form game played in every period $t = 0, 1, \ldots$

1 is a long-run player and 2 is a short-run player (series of one-period players or a very impatient player). 2 plays a best response to 1's anticipated action at every date.

Fudenberg, Kreps, and Maskin (1988): folk theorem if game is common knowledge

- $B_2$: 2's mixed best responses in stage game to 1's mixed actions
- $\underline{u}_1 = \min_{\sigma_2 \in B_2} \max_{a_1 \in A_1} u_1(a_1, \sigma_2)$
- Any payoff for player 1 above $\underline{u}_1$ is sustainable in a subgame perfect equilibrium for high $\delta$

Fudenberg and Levine (1989): if game is perturbed to allow for irrational types of player 1, folk theorem overturned

- $u_1^* = \max_{a_1 \in A_1} \min_{\sigma_2 \in BR_2(a_1)} u_1(a_1, \sigma_2)$: Stackelberg payoff
- 1 obtains his Stackelberg payoff in any Nash equilibrium for high $\delta$

Compare $\underline{u}_1$ and $u_1^*$ for Cournot duopoly.

# Perturbed Game

- $\Omega$: countable space of types for player 1, prior $\mu$
- Only player 1 knows his type
- $u_1(a, \omega)$: player 1's payoff depends on $\omega$; player 2's does not
- $\omega_0$: "rational" type of player 1 with payoffs given by original $u_1$
- $\omega(a_1)$: "crazy" type of player 1 for which playing $a_1$ at every history is a strictly dominant strategy in the repeated game
- $\omega^* = \omega(a_1^*)$ with $\mu(\omega^*) > 0$

# Key Lemma

- Any strategy profile $\sigma$ (together with $\mu$) generates a unique joint distribution over play paths and types $\pi \in \Delta((A_1 \times A_2)^\infty \times \Omega)$
- $h^*$: event in $(A_1 \times A_2)^\infty \times \Omega$ in which $a_1^t = a_1^*$ for all $t$
- $\pi_t^* = \pi(a_1^t = a_1^* | h^{t-1})$: probability of $a_1^*$ at $t$ conditional on history $h^{t-1}$
- $n(\pi_t^* \leq \overline{\pi})$: number of periods $t$ s.t. $\pi_t^* \leq \overline{\pi}$ for $\overline{\pi} \in (0, 1)$
- $\pi_t^*$ and $n(\pi_t^* \leq \overline{\pi})$ are random variables defined on path-type space

**Lemma 1**

*Let $\sigma$ be a strategy profile such that $\pi(h^*|\omega^*) = 1$. Then*

$$\pi\left(n(\pi_t^* \leq \overline{\pi}) \leq \frac{\ln \mu^*}{\ln \overline{\pi}} \,\middle|\, h^*\right) = 1.$$

# Proof

$h^t$: history of length $t$ with $\pi(h^t) > 0$ in which player 1 played $a_1^*$ every period

$h^{t,1}$ ($h^{t,2}$): event that $h^{t-1}$ is observed and player 1 (2) plays at $t$ as in $h^t$

$$\pi(\omega^*|h^t) = \frac{\pi(h^t \,\&\, \omega^*|h^{t-1})}{\pi(h^t|h^{t-1})} = \frac{\pi(\omega^*|h^{t-1})\pi(h^t|\omega^*, h^{t-1})}{\pi(h^t|h^{t-1})}$$

$$= \frac{\pi(\omega^*|h^{t-1})\pi(h^{t,1}|\omega^*, h^{t-1})\pi(h^{t,2}|\omega^*, h^{t-1})}{\pi(h^{t,1}|h^{t-1})\pi(h^{t,2}|h^{t-1})}$$

$$= \frac{\pi(\omega^*|h^{t-1})\pi(h^{t,2}|\omega^*, h^{t-1})}{\pi(h^{t,1}|h^{t-1})\pi(h^{t,2}|h^{t-1})}$$

$$= \frac{\pi(\omega^*|h^{t-1})}{\pi_t^*}$$

# Proof

$$\pi(\omega^*|h^t) = \frac{\pi(\omega^*|h^{t-1})}{\pi_t^*} = \ldots = \frac{\pi(\omega^*|h^0)}{\pi_t^*\pi_{t-1}^* \cdots \pi_0^*} = \frac{\mu^*}{\pi_t^*\pi_{t-1}^* \cdots \pi_0^*}$$

Since $\pi(\omega^*|h^t) \leq 1$, at most $\ln \mu^* / \ln \overline{\pi}$ terms in the denominator of the last expression can be $\leq \overline{\pi}$.

Therefore, with probability 1,

$$n(\pi_t^* \leq \overline{\pi}) \leq \ln \mu^* / \ln \overline{\pi}.$$

# Main Result

- $u_m = \min_{\sigma_2} u_1(a_1^*, \sigma_2, \omega_0)$: lowest stage payoff for 1 when he plays $a_1^*$
- $u_M = \max_a u_1(a, \omega_0)$: highest stage payoff for 1
- $\overline{u}_1 = \max_{a_1} \max_{\sigma_2 \in BR_2(a_1)} u_1(a_1, a_2)$ : "upper" Stackelberg payoff
- $\underline{v}_1(\delta, \mu, \omega_0)$ ($\overline{v}_1(\delta, \mu, \omega_0)$): infimum (supremum) of 1's payoffs in repeated game across Nash equilibria in which 1 uses a pure strategy

## Theorem 1

*For any value $\mu^*$, there exists a number $\kappa(\mu^*)$ s.t. for all $\delta$ and all $(\mu, \Omega)$ with $\mu(\omega^*) = \mu^*$, we have*

$$\underline{v}_1(\delta, \mu, \omega_0) \geq \delta^{\kappa(\mu^*)} u_1^* + (1 - \delta^{\kappa(\mu^*)}) u_m.$$

*Moreover, there exists $\kappa$ such that for all $\delta$, we have*

$$\overline{v}_1(\delta, \mu, \omega_0) \leq \delta^{\kappa} \overline{u}_1 + (1 - \delta^{\kappa}) u_M.$$

*As $\delta \to 1$, the payoff bounds converge to $u_1^*$ and $\overline{u}_1$ (generically identical).*

# Proof

$\exists\,\overline{\pi} < 1$ s.t. in any Nash equilibrium player 2 plays a best response to $a_1^*$ at every stage $t$ where $\pi_t^* > \overline{\pi}$

- Pure strategy best response correspondence has closed graph.
- Action spaces are finite.

$\exists\,\kappa(\mu^*)$ s.t. $\pi(n(\pi^* \leq \overline{\pi}) > \kappa(\mu^*) \mid h^*) = 0$ (by the lemma)

If rational player 1 deviates to playing $a_1^*$ always, there are at most $\kappa(\mu^*)$ periods in which player 2 will not play a best response to $a_1^*$. Then payoff from deviating is at least

$$\delta^{\kappa(\mu^*)} u_1^* + (1 - \delta^{\kappa(\mu^*)}) u_m.$$

Proof for upper bound requires a version of the lemma for $\omega_0$... from the perspective of rational player 1, player 2 plays a best response to his action at all but a finite set of dates.

Fudenberg and Levine (1992): extension to mixed strategy Nash equilibria

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