14.124 Problem Set 4

Question 1

(a) Let t_i be the total tax payment by the monopoly of type i. The program is as follows:

$$\begin{aligned} \max_{t_i, x_i} \quad & \sum_i p_i t_i; \\ s.t. \quad & x_i - c_i x_i - t_i \geq 0, \text{ for all } i; \\ & x_i - c_i x_i - t_i \geq x_j - c_i x_j - t_j, \text{ for all } i, j. \end{aligned}$$

(b) The firm's profit has strictly decreasing differences in c and x, so any implemntable x_i must be weakly decreasing in i. To see this, simply add up the IC constraints that types i and j do not immitate each other, and the t_i and t_j cancel out and we obtain that

$$-c_j x_j - c_i x_i \geq -c_j x_i - c_i x_j.$$

It can be rewritten as $(c_i - c_j)(x_i - x_j) \leq 0$. Therefore, if $i > j$ (so that $c_i > c_j$), then $x_i \leq x_j$.

(c) Since the firm's profit is decreasing in i, the IR constraint is only binding for type n, which means that $t_n = (1 - c_n)x_n$. At optimality type i is indifferent between reporting i and reporting $(i + 1)$, so $t_i - t_{i+1} = (1 - c_i)x_i - (1 - c_i)x_{i+1}$. (Convince yourself this fact if you did not come to recitation.) Therefore,

$$t_i = (1 - c_i)x_i - \sum_{j=i+1}^{n} (c_j - c_{j-1})x_j. \tag{1}$$

The summation is type i's profit. Therefore, the regulator's maximum payoff under a production plan (x_i) is

$$\sum_i p_i \left[(1-c_i)x_i - \sum_{j=i+1}^{n}(c_j - c_{j-1})x_j\right] = \sum_i \left[p_i(1-c_i) - \sum_{k=1}^{i-1} p_k(c_i - c_{i-1})\right] x_i.$$

The constraint is that $0 \leq x_n \leq x_{n-1} \leq ... \leq x_1 \leq 1$. Since the objective function is linear in x_i, each x_i reaches its upper bound (x_{i-1} if $i > 1$ or 1 if $i = 1$) or lower bound (x_{i+1} if $i < n$ and 0 if $i = n$). Therefore, there exists a k such that $x_i = 1$ when $i \geq k$ and $x_i = 1$ if $i > k$. By Eq. (1), $t_i = 0$ for $i > k$ and for $i \leq k$,

$$t_i = 1 - c_i - \sum_{j=i+1}^{k}(c_j - c_{j-1}) = 1 - c_k.$$

Question 2

Notice that $c(q, \beta)$ has strictly increasing differences in q and β, so every non-increasing $q(\beta)$ can be implemented. Indeed $q(\beta) = 1/\beta^2$ is decreasing in β. The envelope theorem implies that the transfer schedule $t(\beta)$ that implements $q(\beta)$ is unique, and is given by

$$t(\beta) = c(q(\beta), \beta) + \pi(\beta_0) - \int_{\beta_0}^{\beta} c_2(q(\tilde{\beta}), \tilde{\beta})d\tilde{\beta},$$

where β_0 is a type, $\pi(\beta_0)$ is a constant, and c_2 is the partial derivative of c with respect to its second argument. Substituting in $q(\beta) = 1/\beta^2$, we obtain that

$$t(\beta) = \frac{1}{\beta} + \pi(\beta_0) - \int_{\beta_0}^{\beta} \frac{1}{\tilde{\beta}^2}d\tilde{\beta} = \frac{2}{\beta} + \pi(\beta_0) - \frac{1}{\beta_0}.$$

Let $A = \pi(\beta_0) - \frac{1}{\beta_0}$. Notice that $\beta = q^{-1/2}$, so

$$p(q) = t(q^{-1/2}) = 2\sqrt{q} + A.$$

Question 3

(a) Since utilities are transferable, the efficient trading rule maximizes the total surplus $(v - c)x$, which means that $x = 1$ if and only if $v \geq c$.

(b) Let $x(m_S, m_B)$ be the trading rule, and $t_S(m_S, m_B)$ and $t_B(m_S, m_B)$ be the transfer rule. Then in a direct mechanism that implements efficient trade, $x(m_S, m_B) = 1$ if $m_B \geq m_S$ and 0 otherwise. The seller's payoff under this mechanism is

$$t_S(m_S, m_B) - cx(m_S, m_B).$$

It is required that $m_S = c$ is optimal for all m_B. Since the seller can choose the m_S that maximizes $t_S(m_S, m_B)$ under the same physical allocation $x(m_S, m_B)$, there exist two functions $t_{S1}(m_B)$ and $t_{S0}(m_B)$ such that $t_S(m_S, m_B) = t_{S,x(m_S,m_B)}(m_B)$. In other words, the payment that the seller receives only depends on the buyer's message and whether trade occurs. IC constraints imply that

$$\begin{aligned} t_{S1}(m_B) - c &\geq t_{S0}(m_B), \text{ if } m_B \geq c; \\ t_{S0}(m_B) &\geq t_{S1}(m_B) - c, \text{ if } m_B < c. \end{aligned}$$

Therefore, $t_{S1}(m_B) - t_{S0}(m_B) = m_B$. Similarly, there exists a function t_{B0} such that $t_B(m_S, m_B) = t_{B0}(m_S)$ when $m_S > m_B$ and $t_B(m_S, m_B) = t_{B0}(m_S) + m_S$ when $m_S \leq m_B$.

(c) This is obvious from the previous part as the requirement forces both t_{S0} and t_{B0} to be zero.

(d) Notice that $t_B(m_S, m_B) - t_S(m_S, m_B) = \min\{m_S - m_B, 0\}$, so the budget breaks whenever $c < v$.

Question 4

(a) Under this mechanism, a buyer's payoff is $(v_b - p)m_b(\hat{v}_b)$ if she reports value $\hat{v}_b$. Clearly reporting the true v_b is optimal. Since the buyer's payoff is always non-negative, she always participates. Similarly, the IC and IR constraints are satisfied for the seller.

(b) The total transfer from the mechanism designer is $-[1-F_b(p)]p+F_s(p)p = [F_s(p)+F_b(p)-1]p$. Now $F_s(p) + F_b(p) = 0$ when $p = -\infty$ and $F_s(p) + F_b(p) = 2$ when $p = \infty$, so there exists a p^* such that

$$F_s(p^*) + F_b(p^*) = 1,$$

which means that the budget is balanced when the price is p^*. The above condition also means that the mass of sellers who trade $(F_s(p^*))$ and the mass of buyers who trade $(1 - F_b(p^*))$ are equal, so the mechanism is feasible.

The efficient trade scheme must satisfy five conditions:

- It is feasible: the mass of buyers who trade equals the mass of sellers who trade;

- If a seller of value v_s trades, all sellers of lower values trade;

- If a buyer of value v_b trades, all buyers of higher values trade;

- If a buyer with value v_b trades and a seller with value v_s trades, then $v_b \geq v_s$;

- If a buyer with value v_b does not trade and a seller with value v_s does not trade, then $v_b \leq v_s$.

The second and the third requirements mean that the efficient scheme is characterized by two thresholds $\bar{v}_s$ and $\underline{v}_b$ such that a seller trades if and only if his value is below $\bar{v}_s$ and a buyer trades if and only if her value is higher than $\underline{v}_b$. The fourth and fifth requirements mean that $\bar{v}_s = \underline{v}_b$. The first requirement implies that $1 - F_b(\underline{v}_b) = F_s(\bar{v}_s)$. Clearly, this condition means that $\bar{v}_s = \underline{v}_b = p^*$.

(c) This mechanism will not be feasible (i.e. sometimes a buyer wants to buy but the seller does not want to sell or the other way around) when there is only one buyer and only one seller.