14.124 Problem Set 2

Question 1

(a) The program is the following:

$$\begin{aligned} \min_{s_i} \quad & \sum_i p_H(i) s_i; \\ s.t. \quad & \sum_i [p_H(i) - p_L(i)] s_i \geq c(H); \\ & s_i \geq 0, \text{ for all } i. \end{aligned}$$

The first constraint is the agent's IC constraint, and the second is the limited liability constraint.

(b) Let μ and ν_i be the Lagrange multipliers of the two constraints, respectively. Then the first-order conditions can be written as follows:

$$1 - \frac{p_H(i) - p_L(i)}{p_H(i)} \mu - \nu_i = 0, \text{ for all } i.$$

Notice that all multipliers are non-negative and a multiplier is zero only if its corresponding constraint is not binding.

Suppose that $\mu = 0$. Then $\nu_i = 1$ for all i which implies that $s_i = 0$ for all i, which cannot satisfy the agent's IC constraint. Therefore, $\mu > 0$. The strict MLRP implies that $(p_H(i) - p_L(i))\mu / p_H(i)$ is strictly increasing in i, which further implies that ν_i is strictly decreasing in i. Therefore, $\nu_i > 0$ and $s_i = 0$ for $i = 1, 2, ..., n-1$. Finally, $s_n > 0$ since otherwise the agent's IC constraint cannot be satisfied.

Question 2

(a) Let p_{NF} be the probability that there is no fire, p_{2e} the probability that damage is 2000 when Adam exerts effort and p_{2ne} the probability that damage is 2000 when Adam does not exert effort. Let x_1, x_2 and x_3 be Adam's consumption in these three cases. Here is the program:

$$
\begin{aligned}
\max_{x_1,x_2,x_3} \quad & p_{NF}u(x_1) + (1-p_{NF})(1-p_{2e})u(x_2) + (1-p_{NF})p_{2e}u(x_3) \\
s.t. \quad & (1-p_{NF})(p_{2ne}-p_{2e})(u(x_2)-u(x_3)) \geq c; \\
& p_{NF}(-x_1) + (1-p_{NF})(1-p_{2e})(-1000-x_2) + (1-p_{NF})p_{2e}(-2000-x_3) \geq -W,
\end{aligned}
$$

where W is Adam's initial wealth.

(b) If Adam's IC constraint is not binding, $x_2 = x_3$ at optimality and the IC constraint is violated. If the company's IR constraint is not binding, then all x's are inifite. Therefore, both constriants are binding. Let $\lambda > 0$ and $\mu > 0$ be Lagrange multipliers of the two constriants. Then the FOCs are

$$
\begin{aligned}
u'(x_1) - \mu &= 0; \\
u'(x_2) + \frac{p_{2ne}-p_{2e}}{1-p_{2e}}\lambda - \mu &= 0; \\
u'(x_3) - \frac{p_{2ne}-p_{2e}}{p_{2e}}\lambda - \mu &= 0.
\end{aligned}
$$

Therefore, $u'(x_2) < u'(x_1) < u'(x_3)$. Since u is strictly concave, u' is strictly decreasing, so $x_3 < x_1 < x_2$. The same ordering is true for the S's. Here is the intuition: we want to encourage Adam to exert the effort, so we need $x_2 > x_3$ to give him incentive. On the other hand, we want to insure him against the risk of fire, so we want to choose x_1 in between.

(c) Since the damage of 2000 does not occur in reality, we can make Adam's payoff as low as possible in that case. Then we set $S_1 = S_2$ and check if the IC constraint is satisfied. If for some reason (such as limited liability) the IC constraint is not satisfied, then $S_1 < S_2$.

Question 3

(a) The agent maximizes

$$\hat{w}(e, s; \alpha, \beta) = \beta + \alpha e - \frac{r}{2}\alpha^2\sigma^2 - \frac{1}{2}(e+s)^2 + s.$$

For $\alpha = 0.3$, $e = 0$ and $s = 1$.

(b) In this case, the agent chooses effort α, and the principal maximizes

$$(1-\alpha)e - \beta = \alpha - \underline{u} - \frac{r}{2}\alpha^2\sigma^2 - \frac{1}{2}\alpha^2,$$

where $\underline{u}$ is the agent's reserved utility. Therefore, the optimal α is $1/(r\sigma^2+1) = 1/9$.

(c) In this case, $\underline{u} = \max_s s - \frac{1}{2}s^2 = \frac{1}{2}$, so

$$\beta = \underline{u} + \frac{r}{2}\alpha^2\sigma^2 - \alpha e + \frac{1}{2}e^2 = \frac{44}{81}.$$

Question 4

(a) If e can be contracted on, the principal can write a contract to maximize total surplus, eliminate risk to the agent, and place the agent exactly at the minimum utility by choosing e and paying only if that level of e is observed. Note that with a risk averse agent and risk neutral principal, optimal risk sharing requires the principal to bear all risk and pay the agent a fixed amount. The principal maximizes

$$\max_{e,s} \mathbb{E}[x|e] - s \quad \text{s.t.} \quad u(s) - c(e) \geq \underline{u}$$

The FOCs give

$$\begin{array}{rcl} 1 & = & \lambda u'(s^*) \\ 1 & = & \lambda c'(e^*) \end{array} \implies u'(e^*) = c'(s^*)$$

This condition together with the participation constraint gives a unique first-best effort e^* and associated payment s^*.

(b) For simplicity, I assume $c(e) \geq 0$, $\forall e$; the problem can be solved without this but it is slightly more tedious. Define $\underline{s}$ such that $u(\underline{s}) = \min\{\underline{u}, u(s^*) - u'(s^*)\}$; by construction $\underline{s} < s^*$. Consider the contract

$$s(x) = \begin{array}{ll} s^* & \text{if} \quad x \in [e^*, e^* + 1] \\ \underline{s} & \quad\quad \text{otherwise} \end{array}$$

When the agent chooses $e = e^*$, then the agent's utility is $\underline{u}$; this is the first-best efficient solution and satisfies IR by construction. All that remains is to verify IC.

If the agent chooses $e > e^*$, then she increases her disutility of effort and lowers her expected payout by placing positive probability on outcomes that pay $\underline{s} < s^*$, therefore she will not choose $e > e^*$. When $e \leq e^* - 1$, the agent's expected utility is at most $\underline{u} - c(e) \leq \underline{u}$ so the agent will not prefer that either. For $e = e^* - \delta$, $\delta \in (0, 1)$, the agent's expected utility is

$$\begin{aligned} \mathbb{E}[U_A|e] &= \mathbb{E}[s(x)|e] - c(e) \\ &\leq (1 - \delta)u(s^*) + \delta u(\underline{s}) - (c(e^*) - \delta c'(e^*)) \\ &= [u(s^*) - c(e^*)] - \delta[(u(s^*) - u(\underline{s}) - c'(e^*)] \\ &\leq \mathbb{E}[U_A|e^*] - \delta[u'(s^*) - c'(e^*)] \\ &= \mathbb{E}[U_A|e^*] \end{aligned}$$

where the first inequality follows from the convexity of $c(e)$. Thus the agent will not prefer e in that region, meaning IC is satisfied and the first-best is achieved.