Chapter 7

Application: Imperfect Competition

Some of the earliest applications of game theory is the analyses of imperfect competition by Cournot (1838) and Bertrand (1883), a century before Nash (1950). This chapter applies the solution concepts of rationalizability and Nash equilibrium to those models of imperfect competition.

7.1 Cournot (Quantity) Competition

Consider n firms. Each firm i produces $q_i \geq 0$ units of a good at marginal cost $c \geq 0$ and sell it at price

$$P = \max\{1 - Q, 0\} \tag{7.1}$$

where

$$Q = q_1 + \cdots q_n \tag{7.2}$$

is the total supply. Each firm maximizes the expected profit. Hence, the payoff of firm i is

$$\pi_i = q_i (P - c). \tag{7.3}$$

Assuming all of the above is commonly known, one can write this as a game in normal form, by setting

- $N = \{1, 2, \ldots, n\}$ as the set of players

- $S_i = [0, \infty)$ as the strategy space of player i, where a typical strategy is the quantity q_i produced by firm i, and
- $\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$ as the payoff function.

Best Response Throughout the course, it will be useful to know the best response of a firm i to the production levels of the other firms. (See also Exercise 3 in Section 4.4.) Write

$$Q_{-i} = \sum_{j \neq i} q_j \tag{7.4}$$

for the total supply of the firms other than firm i. If $Q_{-i} > 1$, then the price $P = 0$ and the best firm i can do is to produce zero and obtain zero profit. Now assume $Q_{-i} \leq 1$. For any $q_i \in (0, 1 - Q_{-i})$, the profit of the firm i is

$$\pi_i (q_i, Q_{-i}) = q_i (1 - q_i - Q_{-i} - c) . \tag{7.5}$$

(The profit is negative if $q_i > 0$.) By setting the derivative of π_i with respect to q_i to zero,[1] one can obtain the best production level

$$q_i^B (Q_{-i}) = \frac{1 - Q_{-i} - c}{2}. \tag{7.6}$$

The profit function is plotted in Figure 7.1. The best response function is plotted in Figure 7.2.

7.1.1 Cournot Duopoly

Now, consider the case of two firms. In that case, $Q_{-i} = q_j$ for $i \neq j$.

Nash Equilibrium Any Nash equilibrium (q_1, q_2) must satisfy

$$q_1 = q_1^B (q_2) \equiv \frac{1 - q_2 - c}{2}$$

and

$$q_2 = q_2^B (q_1) \equiv \frac{1 - q_1 - c}{2}.$$

[1] I.e.

$$\frac{\partial \pi_i}{\partial q_i} = 1 - 2q_i - Q_{-i} - c = 0.$$

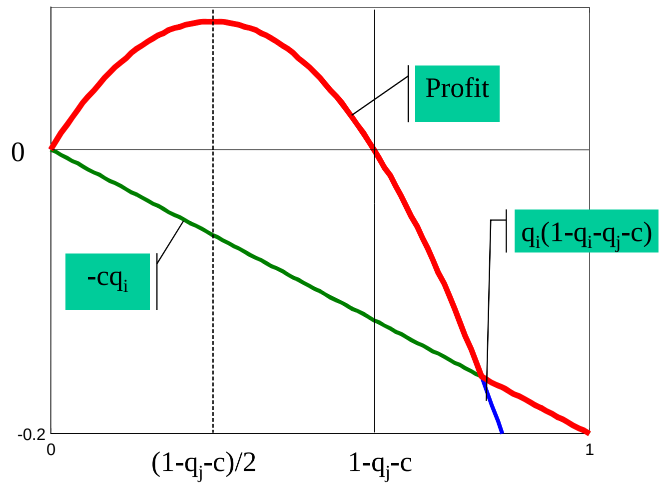


Figure 7.1:

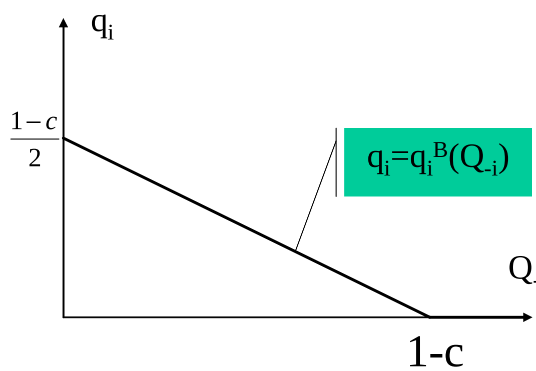


Figure 7.2:

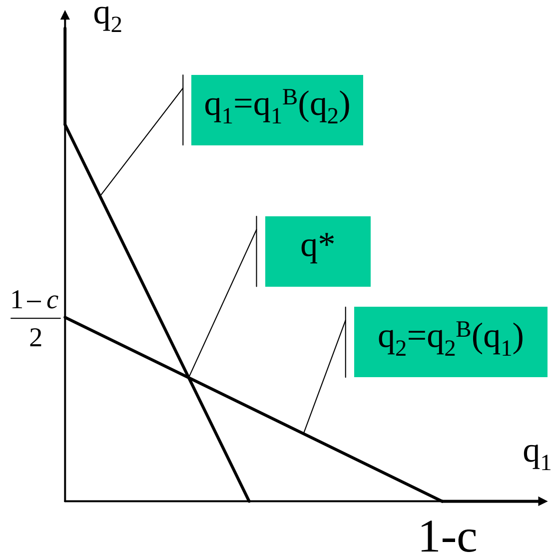


Figure 7.3:

Solving these two equations simultaneously, one can obtain

$$q_1^* = q_2^* = \frac{1-c}{3}$$

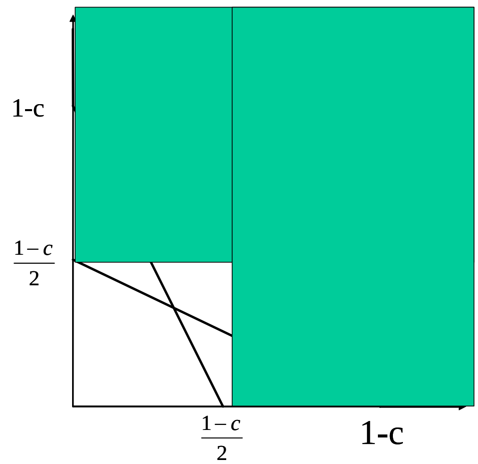
as the only Nash equilibrium. Graphically, as in Figure 7.3, one can plot the best response functions of each firm and identify the intersections of the graphs of these functions as Nash equilibria. In this case, there is a unique intersection, and therefore there is a unique Nash equilibrium.

Rationalizability The (linear) Cournot duopoly game considered here is dominance solvable> That is, there is a unique rationalizable strategy. Let us first consider the first couple rounds of elimination to see this intuitively. I will then show mathematically that this is indeed the case.

Round 1 Notice that a strategy $\hat{q}_i > (1-c)/2$ is strictly dominated by $(1-c)/2$. To see this, consider any q_j. As in Figure 7.1, $\pi_i(q_i, q_j)$ is strictly increasing until $q_i = (1-c-q_j)/2$ and strictly decreasing thereafter. In particular,

$$\pi_i((1-c-q_j)/2, q_j) \geq \pi_i((1-c)/2, q_j) > \pi_i(\hat{q}_i, q_j),$$

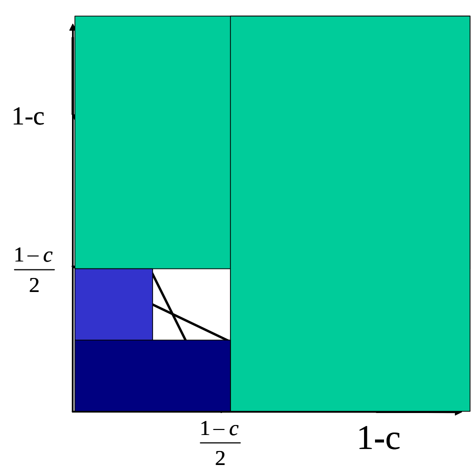
showing that $\hat{q}_i$ is strictly dominated by $(1-c)/2$. We therefore eliminate all $\hat{q}_i > (1-c)/2$ for each player i. The resulting strategies are as follows, where the shaded area is eliminated:



Round 2 In the remaining game $q_j \leq (1-c)/2$. Consequently, any strategy $\hat{q}_i < (1-c)/4$ is strictly dominated by $(1-c)/4$. To see this, take any $q_j \leq (1-c)/2$ and recall from Figure 7.1 that π_i is strictly increasing until $q_i = (1-c-q_j)/2$, which is greater than or equal to $(1-c)/4$. Hence,

$$\pi_i(\hat{q}_i, q_j) < \pi_i((1-c)/4, q_j) \leq \pi_i((1-c-q_j)/2, q_j),$$

showing that $\hat{q}_i$ is strictly dominated by $(1-c)/4$. We will therefore eliminate all $\hat{q}_i$ with $\hat{q}_i < (1-c)/4$. The remaining strategies are as follows:



Notice that the remaining game is a smaller replica of the original game. Applying the same procedure repeatedly, one can eliminate all strategies except for the Nash equilibrium. (After every two rounds, a smaller replica is obtained.) Therefore, the only rationalizable strategy is the unique Nash equilibrium strategy:

$$q_i^* = (1 - c)/3.$$

A more formal treatment One can prove this more formally by invoking the following lemma repeatedly:

Lemma 7.1 *Given that $q_j \leq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i < q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1 - \bar{q} - c)/2$. Given that $q_j \geq \bar{q}$, every strategy $\hat{q}_i$ with $\hat{q}_i > q_i^B(\bar{q})$ is strictly dominated by $q_i^B(\bar{q}) \equiv (1 - \bar{q} - c)/2$.*

Proof. To prove the first statement, take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly increasing in q_i at any $q_i < q_i^B(q_j)$. Since $\hat{q}_i < q_i^B(\bar{q}) \leq q_i^B(q_j)$,[2] this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i\left(q_i^B(\bar{q}), q_j\right).$$

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$.

To prove the second statement, take any $q_j \leq \bar{q}$. Note that $\pi_i(q_i; q_j)$ is strictly decreasing in q_i at any $q_i > q_i^B(q_j)$. Since $q_i^B(q_j) \leq q_i^B(\bar{q}) < \hat{q}_i$, this implies that

$$\pi_i(\hat{q}_i, q_j) < \pi_i\left(q_i^B(\bar{q}), q_j\right).$$

That is, $\hat{q}_i$ is strictly dominated by $q_i^B(\bar{q})$. ■

Now, define a sequence $q^0, q^1, q^2, \ldots$ by $q^0 = 0$ and

$$q^m = q_i^B\left(q^{m-1}\right) \equiv \left(1 - q^{m-1} - c\right)/2 = (1 - c)/2 - q^{m-1}/2$$

[2]This is because q_i^B is decreasing.

for all $m > 0$. That is,

$$\begin{aligned}
q^0 &= 0 \\
q^1 &= \frac{1-c}{2} \\
q^2 &= \frac{1-c}{2} - \frac{1-c}{4} \\
q^3 &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} \\
&\cdots \\
q^m &= \frac{1-c}{2} - \frac{1-c}{4} + \frac{1-c}{8} - \cdots - (-1)^m \frac{1-c}{2^m} \\
&\cdots
\end{aligned}$$

Theorem 7.1 *The set of remaining strategies after any odd round m $(m = 1, 3, \ldots)$ is $[q^{m-1}, q^m]$. The set of remaining strategies after any even round m $(m = 2, 4, \ldots)$ is $[q^m, q^{m-1}]$. The set of rationalizable strategies is $\{(1-c)/3\}$.*

Proof. We use mathematical induction on m. For $m = 1$, we have already proven the statement. Assume that the statement is true for some odd m. Then, for any q_j available at even round $m+1$, we have $q^{m-1} \leq q_j \leq q^m$. Hence, by Lemma 7.1, any $\hat{q}_i < q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. That is, if q_i survives round $m+1$, then $q^{m+1} \leq q_i \leq q^m$. On the other hand, every $q_i \in [q^{m+1}, q^m] = \left[q_i^B(q^m), q_i^B(q^{m-1})\right]$ is a best response to some q_j with $q^{m-1} \leq q_j \leq q^m$, and it is not eliminated. Therefore, the set of strategies that survive the even round $m+1$ is $[q^{m+1}, q^m]$.

Now, assume that the statement is true for some even m. Then, for any q_j available at odd round $m+1$, we have $q^m \leq q_j \leq q^{m-1}$. Hence, by Lemma 7.1, any $\hat{q}_i > q_i^B(q^m) = q^{m+1}$ is strictly dominated by q^{m+1} and eliminated. Moreover, every $q_i \in [q^m, q^{m+1}] = \left[q_i^B(q^{m-1}), q_i^B(q^m)\right]$ is a best response to some q_j with $q^m \leq q_j \leq q^{m-1}$, and it is not eliminated. Therefore, the set of strategies that survive the odd round $m+1$ is $[q^m, q^{m+1}]$.

Finally, notice that

$$\lim_{m\to\infty} q^m = (1-c)/3.$$

Therefore, the intersections of the above intervals is $\{(1-c)/3\}$, which is the set of rationalizable strategies. ■

7.1.2 Cournot Oligopoly

We will now consider the case of three or more firms. When there are three or more firms, rationalizability does not help: one cannot eliminate any strategy less than the monopoly production $q^1 = (1-c)/2$.

Rationalizability In the first round, one can eliminate any strategy $q_i > (1-c)/2$, using the same argument in the case of duopoly. But in the second round, the maximum possible total supply by the other firms is

$$(n-1)(1-c)/2 \geq 1-c,$$

where n is the number of firms. The best response to this aggregate supply level is 0. Hence, one cannot eliminate any strategy in round 2. The elimination process stops, yielding $[0,(1-c)/2]$ as the set of rationalizable strategies. Since the set of rationalizable strategies is large, rationalizability has a weak predictive power in this game.

Nash Equilibrium While rationalizability has a weak predictive power in that the set of rationalizable strategies is large, Nash equilibrium remains to have a strong predictive power. There is a unique Nash equilibrium. Recall that $q^* = (q_1^*, q_2^*, \ldots, q_n^*)$ is a Nash equilibrium if and only if

$$q_i^* = q_i^B\left(\sum_{j\neq i} q_j^*\right) = \frac{1-\sum_{j\neq i} q_j^* - c}{2}$$

for all i, where the second equality by (7.6) and the fact that in equilibrium the firms cannot have negative profits in equilibrium (i.e. $\sum_{j\neq i} q_j^* \leq 1-c$). Rewrite this equation system more explicitly:

$$\begin{aligned}
2q_1^* + q_2^* + \cdots + q_n^* &= 1-c \\
q_1^* + 2q_2^* + \cdots + q_n^* &= 1-c \\
&\ \ \vdots \\
q_1^* + q_2^* + \cdots + 2q_n^* &= 1-c.
\end{aligned}$$

For any i and j, by subtracting jth equation from ith, one can obtain

$$q_i^* - q_j^* = 0.$$

Hence,

$$q_1^* = q_2^* = \cdots = q_n^*.$$

Substituting this into the first equation, one then obtains

$$(n+1)\, q_1^* = 1 - c;$$

i.e.

$$q_1^* = q_2^* = \cdots = q_n^* = \frac{1-c}{n+1}.$$

Therefore, there is a unique Nash equilibrium, in which each firm produces $(1-c)\,/\,(n+1)$.

In the unique equilibrium, the total supply is

$$Q = \frac{n}{n+1}\,(1-c)$$

and the price is

$$P = c + \frac{1-c}{n+1}.$$

The profit level for each firm is

$$\pi = \left(\frac{1-c}{n+1}\right)^2.$$

As n goes to infinity, the total supply Q converges to $1-c$, and price P converges to c. These are the values at which the demand ($P = \max\{1-Q, 0\}$) is equal to supply ($P = c$), which is called (perfectly) competitive equilibrium. When there are few firms, however, the price is significantly higher than the competitive price c, and the total supply is significantly lower than the competitive supply $1-c$. We will next consider another model, in which two firms are enough for the competitive outcome.

7.2 Bertrand (Price) Competition

Consider two firms. Simultaneously, each firm i sets a price p_i. The firm i with the lower price $p_i < p_j$ sells $1-p_i$ units and the other firm cannot sell any. If the firms set the same price, the demand is divided between them equally. That is, the amount of sales for firm i is

$$Q_i\,(p_1, p_2) = \begin{cases} 1 - p_i & \text{if } p_i < p_j \\ \frac{1-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Assume that it costs noting to produce the good (i.e. $c = 0$). Therefore, the profit of a firm i is

$$\pi_i(p_1, p_2) = p_i Q_i(p_1, p_2) = \begin{cases} (1-p_i)\, p_i & \text{if } p_i < p_j \\ \frac{(1-p_i)p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{otherwise.} \end{cases}$$

Assuming all of the above is commonly known, one can write this formally as a game in normal form by setting

- $N = \{1, 2\}$ as the set of players
- $S_i = [0, \infty)$ as the set of strategies for each i, with price p_i a typical strategy,
- π_i as the utility function.

Observe that when $p_j = 0$, $\pi_i(p_1, p_2) = 0$ for every p_i, and hence every p_i is a best response to $p_j = 0$. This has two important implications:

1. Every strategy is rationalizable (one cannot eliminate any strategy because each of them is a best reply to zero).
2. $p_1^* = p_2^* = 0$ is a Nash equilibrium.

In the rest of the notes, I will first show that this is indeed the only Nash equilibrium. In other words, even with two firms, when the firms compete by setting prices, the competitive equilibrium will emerge. I will then show that if we modify the game slightly by discretizing the set of allowable prices and putting a minimum price, then the game becomes dominance-solvable, i.e., only one strategy remains rationalizable. In the modified game, the minimum price is the only rationalizable strategy, as in competitive equilibrium. Finally I will introduce small search costs on the part of consumers, who are not modeled as players, and illustrate that the equilibrium behavior is dramatically different from the equilibrium behavior in the original game and competitive equilibrium.

7.2.1 Nash Equilibrium

Theorem 7.2 *In Bertrand competition, the only Nash equilibrium is $p^* = (0, 0)$.*

Proof. We have seen already that $p^* = (0,0)$ is a Nash equilibrium. I will here show that if (p_1, p_2) is a Nash equilibrium, then $p_1 = p_2 = 0$. To do this, take any Nash equilibrium (p_1, p_2). I first show that $p_1 = p_2$. Towards a contradiction, suppose that $p_i > p_j$. If $p_j = 0$, then $\pi_j(p_i, p_j) = 0$, while $\pi_j(p_i, p_i) = (1 - p_i) p_i/2 > 0$. That is, choosing p_i is a profitable deviation for firm j, showing that $p_i > p_j = 0$ is not a Nash equilibrium. Therefore, in order $p_i > p_j$ to be an equilibrium, it must be that $p_j > 0$. But then, firm i has a profitable deviation: $\pi_i(p_i, p_j) = 0$ while $\pi_i(p_j, p_{ij}) = (1 - p_j) p_j/2 > 0$. All in all, this shows that one cannot have $p_i > p_j$ in equilibrium. Therefore, $p_1 = p_2$. But if $p_1 = p_2$ in a Nash equilibrium, then it must be that $p_1 = p_2 = 0$. This is because if $p_1 = p_2 > 0$, then Firm 1 would have a profitable deviation: $\pi_1(p_1, p_2) = (1 - p_1) p_1/2$ while $\pi_1(p_1 - \varepsilon, p_2) = (1 - p_1 + \varepsilon)(p_1 - \varepsilon)$, which is close to $(1 - p_1) p_1$ when ε is close to zero. ■

A graphical proof for the above result is as follows. Recall that (p_1, p_2) is a Nash equilibrium if and only if it is in the intersection of the best responses. Recall also from Exercise 3.e of Section 4.4 that everything is a best response to $p_j = 0$ and nothing is a best response to any $p_j = 0$. Hence, as shown in Figure **??**, the best responses intersect each other only at $(0,0)$, showing that $(0,0)$ is the only Nash equilibrium.

7.2.2 Rationalizability with discrete prices

Now suppose that the firms have to set prices as multiples of pennies, and they cannot charge zero price. That is, the set of allowable prices is

$$P = \{0.01, 0.02, 0.03, \ldots\}.$$

The important assumption here is that the minimum allowable price $p^{\min} = 0.01$ yields a positive profit. We will now see that the game is "dominance-solvable" under this assumption. In particular $p^{\min}$ is the only rationalizable strategy, and it is the only Nash equilibrium strategy. Let us start with the first step.

Step 1: *Any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{\min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.*

Proof. Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon(p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon\left(p^{\min}\right) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j. We have

$$\pi_i(p_i, p_j) \leq p_i Q(p_i) = p_i(1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand,

$$\begin{aligned}
\pi_i(\sigma^\epsilon, p_j) &= (1-\epsilon) p^{mon}(1 - p^{mon}) + \epsilon p^{\min}\left(1 - p^{\min}\right) \\
&> (1-\epsilon) p^{mon}(1 - p^{mon}) \\
&= 0.25(1-\epsilon).
\end{aligned}$$

Thus, $\pi_i(\sigma^\epsilon, p_j) > 0.2499 \geq \pi_i(p_i, p_j)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i(p_i, p_j) = 0$. On the other hand,

$$\pi_i(\sigma^\epsilon, p_j) = (1-\epsilon) p^{mon}(1 - p^{mon})/2 + \epsilon p^{\min}\left(1 - p^{\min}\right)$$

when $p_j = p^{mon}$, and

$$\pi_i(\sigma^\epsilon, p_j) = \epsilon p^{\min}\left(1 - p^{\min}\right)$$

when $p_j < p^{mon}$. In either case,

$$\pi_i(\sigma^\epsilon, p_j) \geq \epsilon p^{\min}\left(1 - p^{\min}\right) > 0 = \pi_i(p_i, p_j).$$

Therefore, σ^ϵ strictly dominates p_i. ■

Step 1 yields the eliminations in the first round 1.

Round 1 By Step 1, all strategies p_i with $p_i > p^{mon} = 0.5$ are eliminated. Moreover, each $p_i \leq p^{mon}$ is a best reply to $p_j = p_i + 1$, and is not eliminated. Therefore, the set of remaining strategies is

$$P^2 = \{0.01, 0.02, \ldots, 0.5\}.$$

Round m Suppose that the set of remaining strategies to round m is

$$P^m = \{0.01, 0.02, \ldots, \bar{p}\}.$$

Then, the strategy $\bar{p}$ is strictly dominated by a mixed strictly dominated by the mixed strategy σ^ϵ with $\sigma^\epsilon(\bar{p} - 0.01) = 1 - \epsilon$ and $\sigma^\epsilon\left(p^{\min}\right) = \epsilon$, as we will see momentarily. We then eliminate the strategy $\bar{p}$. There will be no more elimination because each $p_i < \bar{p}$ is a best reply to $p_j = p_i + 0.01$.

To prove that $\bar{p}$ is strictly dominated by σ^ϵ, note that the profit from $\bar{p}$ for player i is

$$\pi_i(\bar{p}, p_j) = \begin{cases} \bar{p}(1-\bar{p})/2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \pi_i\left(\sigma^\epsilon_{\bar{p}}, \bar{p}\right) &= (1-\epsilon)(\bar{p}-0.01)(1-\bar{p}+0.01) + \epsilon p^{\min}\left(1-p^{\min}\right) \\ &> (1-\epsilon)(\bar{p}-0.01)(1-\bar{p}+0.01) \\ &= (1-\epsilon)\left[\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})\right]. \end{aligned}$$

Then, $\pi_i(\sigma^\epsilon, \bar{p}) > \pi_i(\bar{p}, p_j)$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})}.$$

But $\bar{p} \geq 0.02$, hence $0.01(1-2\bar{p}) < \bar{p}(1-\bar{p})/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}(1-\bar{p})/2}{\bar{p}(1-\bar{p}) - 0.01(1-2\bar{p})} > 0$$

so that $\pi_i\left(\sigma^\epsilon_{\bar{p}}, \bar{p}\right) > \pi_i(\bar{p}, p_j)$. Moreover, for any $p_j < \bar{p}$,

$$\begin{aligned} \pi_i\left(\sigma^\epsilon_{\bar{p}}, p_j\right) &= (1-\epsilon)(\bar{p}-0.01)(1-\bar{p}+0.01) + \epsilon p^{\min}\left(1-p^{\min}\right) \\ &\geq \epsilon p^{\min}\left(1-p^{\min}\right) > 0 = \pi_i(\bar{p}, p_j), \end{aligned}$$

showing that $\sigma^\epsilon_{\bar{p}}$ strictly dominates $\bar{p}$, and completing the proof.

Therefore, the process continues until the set of remaining strategies is $\left\{p^{\min}\right\}$ and it stops there. Therefore, $p^{\min}$ is the only rationalizable strategy.

Since players can put positive probability only on rationalizable strategies in a Nash equilibrium, the only possible Nash equilibrium is $\left(p^{\min}, p^{\min}\right)$, which is clearly a Nash equilibrium.

7.2.3 Price competition with search costs

This section illustrates that the equilibrium behavior is quite different when the consumers need to engage a costly search in order to find the prices offered by the firms, regardless of how small these costs are.

For simplicity, allow only two prices: 3 and 5. Suppose that the demand for the good comes from a single buyer, for who the value of the good is 6. She needs only 1 unit of good. Unlike before the buyer has a very small search cost $c_s \in (0,1)$. She can check the prices by paying c_s.

The game is as follows:

- The two firms set prices $p_1 \in \{3,5\}$ and $p_2 \in \{3,5\}$ and the consumer decides whether to check the prices, all simultaneously.

- If she checks the prices, then she buys from the firm with the lower price. If she decides not to check or if $p_1 = p_2$, then she buys from either of the firms with equal probabilities. This behavior is set, so that the strategies of the consumer is only "check" and "no check".

Formally,

- $N = \{1, 2, B\}$ is the set of players;

- $S_1 = S_2 = \{3,5\}$ and $S_B = \{\text{check, no check}\}$ are the strategy sets; and

- the payoffs are as in the following table:

check

1\2	5	3
5	$5/2, 5/2, 1 - c_s$	$0, 3, 3 - c_s$
3	$3, 0, 3 - c_s$	$3/2, 3/2, 3 - c_s$

no check

1\2	5	3
5	$5/2, 5/2, 1$	$5/2, 3/2, 2$
3	$3/2, 5/2, 2$	$3/2, 3/2, 3$

Here, the first entry is the payoff of firm 1, the second entry is the payoff of firm 2, and the final entry is the payoff of the buyer. Firm 1 chooses the row; firm 2 chooses the column, and the buyer chooses the matrix. We computed the payoffs, following the set behavior above. For example, if the consumer doesn't check the price, he buys from the either firm with probability 0.5. Hence, the payoff of firm i is $p_i/2$, independent of p_j. The payoff of the buyer is

$$0.5\,(6 - p_1) + 0.5\,(6 - p_2) = 6 - \frac{p_1 + p_2}{2}.$$

If the buyer checks and $p_1 = p_2$, then the payoffs are: $p_1/2$ to each firm and $6 - p_1 - c_s$ to the buyer. If the buyer checks and $p_i < p_j$, then the buyer buys one unit from i, and the payoff of firm i is p_i; the payoff of firm j is 0, and the payoff of the buyer is $6 - p_i - c_s$.

A quick glance at the above table reveals that the only pure strategy Nash equilibrium is the both firm set price to 5 ($p_1 = p_2 = 5$), and the buyer does not check the prices. This is clearly different from the previous games, where price competition pushes the prices to the minimum.

It is easy to check that ($p_1 = p_2 = 5;$ no check) is a Nash equilibrium: Given "no check", $p_i = 5$ dominates $p_i = 3$. Given that prices are equal, the buyer saves c_s by not checking.

It is also easy to check that this is the only Nash equilibrium in pure strategies. If $p_1 = p_2$, the best response of the buyer is "no check". If buyer doesn't check, then the best reply of each firm is 5. Therefore, the only equilibrium with $p_1 = p_2$ is ($p_1 = p_2 = 5;$ no check). On the other hand, there cannot be a Nash equilibrium with $p_1 = p_2$. To see this, suppose that $p_i = 5$ and $p_j = 3$. Then, the buyer gets $3 - c_s$ when she checks and 2 when she does not. The best reply is to check because $c_s < 1$. That is, ($p_i = 5$, $p_j = 3$, no check) is not an equilibrium. In order to have an equilibrium, she must check. But in that case, j gets 0. Firm j could get the higher payoff of $5/2$ by setting $p_j = 5$. Therefore, ($p_i = 5$, $p_j = 3$, check) is not an equilibrium either.

There is also a symmetric Nash equilibrium in mixed strategies. To find the equilibrium, let us write q for the probability that a firm sets $p_i = 5$ (the probabilities are equal by assumption) and r for the probability that buyer checks. The expected payoff from checking for the buyer is

$$U_B\,(\text{check}; q) = q^2 + 3\left(1 - q^2\right) - c_s = 3 - 2q^2 - c_s.$$

This is because the buyer gets $1 - c_s$, when $p_1 = p_2 = 5$, which happens with probability q^2, and $3 - c_s$ otherwise (with probability $1 - q^2$). If she doesn't check her expected payoff is

$$U_B\,(\text{no check}; q) = q + 3\,(1 - q) = 3 - 2q.$$

(Since she chooses the firm randomly without knowing the prices, the probability that the price will be high is q.) We are looking for a mixed strategy Nash equilibrium with

$0 < r < 1$. In that case, the buyer must be indifferent:

$$\begin{aligned} U_B(\text{check}; q) &= U_B(\text{no check}; q) \\ 2q(1-q) &= c_s. \end{aligned} \tag{7.7}$$

That is,

$$q = \frac{1 \mp \sqrt{1-2c_s}}{2}$$

On the other hand, given that the buyer checks with probability r and the other firm charges high price with probability q, the payoff from $p_i = 5$ is

$$U_i(5; q, r) = (1 - r(1-q))\, 5/2.$$

This is because the firm cannot sell if the buyer checks (r) and the other firm charges low price $(1-q)$; otherwise he will sell with probability 0.5. In that case, the payoff from $p_i = 3$ is

$$U_i(3; q, r) = 3qr + (1-qr)\, 3/2.$$

This is because the firm will sell with probability 1, getting the payoff of 3, if the buyer checks and the other firm sets a high price; otherwise the firm gets 3/2. Since $q \in (0,1)$, the firm must be indifferent:

$$\begin{aligned} U_i(5; q, r) &= U_i(3; q, r) \\ (1 - r(1-q))\, 5/2 &= 3qr + (1-qr)\, 3/2. \end{aligned}$$

That is,

$$r = \frac{2}{5-2q}.$$

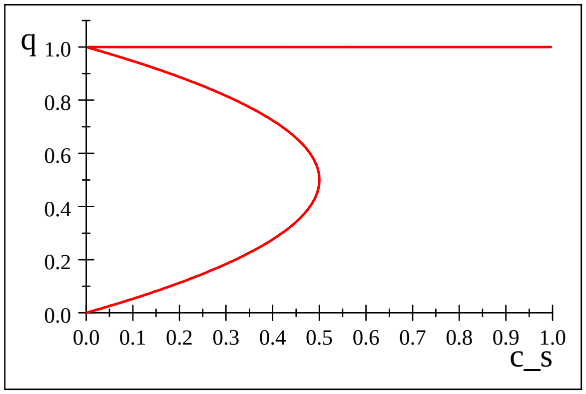
There are two symmetric mixed strategy equilibria:

$$\left(q = \frac{1+\sqrt{1-2c_s}}{2}, r = \frac{2}{4-\sqrt{1-2c_s}}\right)$$

and

$$\left(q = \frac{1-\sqrt{1-2c_s}}{2}, r = \frac{2}{4+\sqrt{1-2c_s}}\right).$$

It is illustrative to plot the possible values of q as a function of c_s, including the pure strategy Nash equilibrium where $q = 1$.



When $c_s > 1/2$ there is a unique Nash equilibrium, in which the firms charge high prices. When $c_s = 1/2$, the there is also a mixed strategy Nash equilibrium, in which the firms charge high and low prices with equal probabilities. As we decrease the cost c_s further, we have two mixed strategy equilibria and a pure strategy equilibria, where the price is high. The probability of charging a high price reacts to the changes in c_s differently in the two equilibria. In one equilibrium, as we decrease c_s to zero, the probability of charging a high price also decreases to zero, when the firms charge low prices. In the other equilibrium, that probability increases to 1, when the firms charge high prices.

7.3 Exercises with Solutions

1. [Homework 2, 2001] Compute the pure-strategy Nash equilibria in the following linear Cournot oligopoly for arbitrary n firms: each firm has marginal cost $c \in (0,1)$ and a fixed cost $F > 0$, which it needs to incur only if it produces a positive amount; the inverse-demand function is given by $P(Q) = \max\{1 - Q, 0\}$, where Q is the total supply.

Solution: Suppose that $m \leq n$ firms produce some positive quantity and the remaining firms produce 0. For any i with positive production q_i^*,

$$q_i^* = \frac{1 - c - \sum_{j \neq i} q_j^*}{2}.$$

As in the usual Cournot model above, the uniqe solution to this equation system is

$$q_i^* = \frac{1 - c}{m + 1}$$

for each firm with positive production. The payoff is

$$\pi^* = \left(\frac{1-c}{m+1}\right)^2 - F.$$

Of course, the firm also has the option of not producing at all and obtaining zero profit. Hence, $\pi^* \geq 0$, i.e.,

$$m \leq \frac{1-c}{\sqrt{F}} - 1 \equiv m^*.$$

Moreover, if $m < n$, the firm that produce 0 must not profit from deviation to positive production:

$$\left(\frac{1-c-mq_i^*}{2}\right)^2 - F \leq 0$$

Since $1-c-mq_i^* = (1-c)/(m+1)$, this simplifies to

$$m \geq m^*/2.$$

Hence, the set of Nash equilibria is as follows. For any integer $m \in [m^*/2, \ldots, m^*]$, m firms produce $(1-c)/(m+1)$ each, and the remaining firms produce 0.

7.4 Exercises

1. [Midterm 1, 2007] Consider the Cournot duopoly with linear demand function $P = 1 - Q$, where P is the price and $Q = q_1 + q_2$ is the total supply.[3] Firm 1 has zero marginal cost. Firm 2 has marginal cost $c(q_2) = q_2$, so that the total cost of producing q_2 is $q_2^2/2$.

   (a) (10 points) Compute all the Nash equilibria.

   (b) (15 points) Compute the set of all rationalizable strategies. Explain your steps.

2. Show that all Nash equilibria of Cournot oligopoly game above are in pure strategies (i.e., one does not need to check for mixed strategy equilibria). (See exercise 17 in Section 6.6.)

3. Can you find a mixed-strategy Nash equilibrium in the Bertrand game above?

[3] Recall that in Cournot duopoly Firms 1 and 2 simultaneously produce q_1 and q_2, and they sell at price P.

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Fall 2012