14.12 Game Theory
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Midterm 2

solutions

Problem 1

a) If we substitute in the value $\theta = -1$, the payoff matrix becomes:

	A	B	C
X	$6,0$	$0,0$	$0,1$
Y	$0,0$	$2,-1$	$9,1$

It is easy to see that strategy C for player 2 dominates all the other strategies. Once we eliminate A and B, then strategy X for player 1 will become conditionally dominated. Therefore, the unique Nash equilibrium of this game is (Y, C). In class we have seen a theorem which states that whenever a stage game has a *unique* Nash equilibrium and it is repeated a *finite* number of times, then there is only one subgame perfect equilibrium with the players playing the Nash equilibrium in every period.

b) Let's now substitute in the value $\theta = 2$. The stage game becomes:

	A	B	C
X	$6,6$	$0,0$	$0,-2$
Y	$0,0$	$2,2$	$9,-2$

We can immediately see that C is dominated for player 2, thus we can safely ignore it when computing the Nash equilibria of the game. There are 2 pure-strategy Nash equilibria (X, A) and (Y, B) and one mixed-strategy Nash equilibria $(\sigma_1, \sigma_2) = \left(\frac{1}{4}X + \frac{3}{4}Y, \frac{1}{4}A + \frac{3}{4}B\right)$. The payoffs associated with the latter equilibrium are $\left(\frac{3}{2}, \frac{3}{2}\right)$, which are lower than those associated with the two pure-strategy Nash equilibria. Thus we can try to use the usual trick of rewarding the players when they comply with the proposed strategy and punish them in case of deviation. The reward will be the most profitable Nash equilibrium (X, A) while the punishment will be the mixed-strategy equilibrium (σ_1, σ_2).

1. Consider the strategies:

$$s_1 = \begin{cases} \text{play } Y \text{ at } t = 1 \\ \text{play } X \text{ at } t = 2 \text{ if } (Y, C) \\ \text{play } \sigma_1 \text{ otherwise} \end{cases}$$

and

$$s_2 = \begin{cases} \text{play } C \text{ at } t = 1 \\ \text{play } A \text{ at } t = 2 \text{ if } (Y, C) \\ \text{play } \sigma_2 \text{ otherwise} \end{cases}$$

The only player who may have an incentive to deviate is player 2 (player 1 is getting the highest payoff at $t = 1$). If player 2 deviates, he will play B and get 2 in the first period and $\frac{3}{2}$ in the second period. His total payoff from deviating will be $\frac{7}{2}$ which is lower than the total payoff of 4 he gets from not deviating. Thus, (Y, C) can be played in the first round of a SPE.

2. Consider the strategies:

$$s_1 = \begin{cases} \text{play } X \text{ at } t = 1 \\ \text{play } X \text{ at } t = 2 \text{ if } (X, B) \\ \text{play } \sigma_1 \text{ otherwise} \end{cases}$$

and

$$s_2 = \begin{cases} \text{play } B \text{ at } t = 1 \\ \text{play } A \text{ at } t = 2 \text{ if } (X, B) \\ \text{play } \sigma_2 \text{ otherwise} \end{cases}$$

Also in this case player 2 is the player with the most profitable deviation. Indeed he can deviate by getting 6 instead of 0 in the first round. Therefore, his total payoff from deviating will be $\frac{15}{2}$ which is higher than the total payoff of 6 he gets from not deviating. Thus, (Y, C) cannot be played in the first round of a SPE.

3. Consider the strategies:

$$s_1 = \begin{cases} \text{play } X \text{ at } t = 1 \\ \text{play } X \text{ at } t = 2 \text{ if } (X, B) \\ \text{play } \sigma_1 \text{ otherwise} \\ \text{play } X \text{ at } t = 3 \text{ if } (X, B), (X, A) \\ \text{play } \sigma_1 \text{ otherwise} \end{cases}$$

and

$$s_2 = \begin{cases} \text{play } C \text{ at } t = 1 \\ \text{play } A \text{ at } t = 2 \text{ if } (X, C) \\ \text{play } \sigma_2 \text{ otherwise} \\ \text{play } A \text{ at } t = 3 \text{ if } (X, C), (X, A) \\ \text{play } \sigma_2 \text{ otherwise} \end{cases}$$

The player with the most profitable deviation is now player 1. She can deviate in period 1 by getting 9 and then $\frac{3}{2}$ in each of the following periods. Her total payoff from deviating will be 12 which is therefore equal to the total payoff of 12 she gets from not deviating. Thus, (X, C) can be played in the first round of a SPE.

c) From part a) we know that whenever $\theta = -1$, C is a dominant strategy for player 2. Thus, in any Bayesian Nash Equilibrium type $\theta = -1$ of player 2 will always play his dominant strategy, that is, $s_2^*(-1) = C$. Let's now consider player 1. Given $s_2^*(-1) = C$, player 1 will get at most $\frac{1}{2}6 + \frac{1}{2}0 = 3$ by playing X (which happens when type $\theta = 2$ plays A) and at least $\frac{1}{2}0 + \frac{1}{2}9 = 4.5$ by playing Y (which happens when type $\theta = 2$ plays A). In other words, given $s_2^*(-1) = C$, X is dominated by Y for player 1.

Let's now consider type $\theta = 2$ of player 2. As shown already above, for this type C is a dominated action. Thus type $\theta = 2$ will choose either A or B (or he will randomize between the two). Given $s_1^* = Y$ and $s_2^*(-1) = C$, type $\theta = 2$ will face the following choice:

$$u_2(s_1^*, s_2^*(-1), s_2, \theta = 2) = \begin{cases} 0 \text{ if } s_2 = A \\ 2 \text{ if } s_2 = B \end{cases}$$

Thus type $\theta = 2$ will choose B. Therefore $(s_1^*, s_2^*(-1), s_2^*(2)) = (Y, C, B)$ is the unique BNE of this game.

Problem 2

a) To be a subgame perfect equilibrium, neither of the players must have a single deviation that could make them better off, given any possible history. Therefore, to show that this is not SPE, we must find a history and a deviation that will make one of the player's better off. The key lies in Bob's acceptance strategy. The strategy says that Bob will accept iff $x \geq 1$. So he will reject anything less than 1. This is not SPE. Suppose Alice offers $x \in (P_S + P_R, 1)$. Let's check the single deviation principle. If he accepts, he will get x. If he rejects, with probability $P_S + P_R$ he will get 1 in the next round (whether he or Alice offers), so he will get a payoff of $P_S + P_R$. Since $x > P_S + P_R$, he will be strictly better off deviating by accepting Alice's offer.

b) To find the SPE, we will look at their acceptance strategies and apply the single deviation principle:
Alice will accept iff $x \geq x_A$. Suppose Bob offers some $\hat{x} < x_A$. If Alice were to deviate and to accept, she will get $\hat{x}$. If she rejects (and then strategies are played as called for), there will be a P_S probability that Bob will offer again and she'll get x_A, there will be a P_R probability that she will offer and get $1 - x_B$, and there will be a P_E probability that she will get 0. So her payoff, if she rejects would be $P_S x_A + P_R(1 - x_B)$. Therefore, for her NOT to have an incentive to deviate, it must be the case that:

$$\hat{x} \leq P_S x_A + P_R(1 - x_B) \;\; \forall \hat{x} < x_A$$

Now consider an $\hat{x} > x_A$. If Alice deviates and rejects this offer, she will get $P_S x_A + P_R(1 - x_B)$. If she accepts it as she is supposed to in the strategy, she will get $\hat{x}$. Therefore, for her NOT to have an incentive to deviate, it must be the case that:

$$\hat{x} \geq P_S x_A + P_R(1 - x_B) \text{ for all } \hat{x} \geq x_A$$

Therefore, the inequality must bind for $\hat{x} = x_A$. So, we get the equation:

$$x_A = P_S x_A + P_R(1 - x_B)$$

Parallel analysis for Bob's acceptance strategy will yield the equation:

$$x_A = P_S x_B + P_R(1 - x_A)$$

Solving these two equations and two unknowns will yield:

$$x_A = x_B = \frac{P_R}{1 + P_R - P_S}$$

Thus, the SPE of the game will be the strategies as listed in the problem where x_A and x_B are as in the above equation. We have calculated it by checking the single deviation principle for both of their acceptance strategies. (You should therefore get 15 points up to this point). The final thing that we need to do is to check, using the single deviation strategies, that neither of them have an incentive to deviate when offering. Alice is suppose to offer x_B to Bob. If she does this, he will accept and she will get a payoff of $1 - x_B$. If she deviates by offering more than x_B to Bob, he will accept (given his strategies), and she will do strictly worse off. Suppose she deviates by offering $x < x_B$. Then Bob will reject, and she will get an expected payoff of $P_S(1 - x_B) + P_R x_A$. She will not have an incentive to deviate as long as: $(1 - x_B) \geq P_S(1 - x_B) + P_R x_A$. Plug in the values of x_A and x_B to see that this is equivalent to the condition that $\frac{1-P_S}{1+P_R-P_S} \geq \frac{P_S-P_S^2+P_R^2}{1+P_R-P_S}$. So we just need to check that: $1 - P_S \geq P_S - P_S^2 + P_R^2$. Rearranging, we get that this would mean that $1 - 2P_S + P_S^2 - P_R^2 \geq 0$. Which can be changed to: $(1 - P_S)^2 - P_R^2 \geq 0$, or $(1 - P_S + P_R)(1 - P_S - P_R) \geq 0$. Since the first factor is clearly greater than 0, and the second factor is equal to $P_E > 0$, this must be true. Therefore, Alice will not have an incentive to deviate when she is offering x_B to Bob. With the exact same analysis, Bob will not have an incentive to deviate when offering x_A to Alice.

Problem 3

1. (a)
   - If the entrant exits, $q_I^* = \frac{1}{2}$. (-2 points for not noting this as part of SPE).
   - If the entrant enters, k is a sunk cost, so this is strategically equivalent to a standard Cournot game, and, from class, $q_I^* = q_E^* = \frac{1}{3}$.
   - Entrant enters, since $\frac{1}{3} \cdot (1 - \frac{2}{3}) - .1 = \frac{1}{9} - .1 > 0$, so higher profits from entering than exiting.

   (b)
   - There are several possible subgames we need to check. When I say "deviation" here, I mean a deviation away from $q = \frac{1}{4}$.
     - First, note that all subgames which occur after a deviation satisfy the single deviation principle, since, by (a), the strategies in these history are repeated SPE of the stage game.
     - Second, in no-deviation histories, note that the entrant never wants to exit: this doesn't affect the future, and it lowers his profit in the present period.
     - Third, in a no-deviation history where the entrant enters, the single deviation principle requires

       $$\max_{q_E} \left(q_E(1 - \frac{1}{4} - q_E) - k \right) + \frac{\delta}{1-\delta}(\frac{1}{9} - k) \leq \left(\frac{1}{8} - k \right) + \frac{\delta}{1-\delta}(\frac{1}{8} - k)$$

       and, since the k's fall out of this equation, the same condition has to hold for the incumbent. Solving the max gives $q_E = 3/8$, so the condition reduces to

       $$\frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9} \leq \frac{1}{8} - \frac{\delta}{1-\delta}\frac{1}{8}$$

       or

       $$\frac{\delta}{1-\delta} \geq \frac{72}{64}.$$

       Solving for the critical δ gives $\widehat{\delta} = \frac{9}{17}$. (8 points for getting this far)
     - Fourth, consider the nodes when there has been no deviation (away from $q = \frac{1}{4}$) but after an "exit" by the Encumbent. According to the proposed strategies, the incumbent is supposed to produce $q_I = \frac{1}{4}$. He'd prefer to produce $q_I = \frac{1}{2}$, but this will lead to a deviation history. He's willing to produce $\frac{1}{4}$ precisely when:

       $$\frac{1}{4} + \frac{\delta}{1-\delta}(\frac{1}{9}) \leq \frac{3}{16} + \frac{\delta}{1-\delta}(\frac{1}{8})$$

       Solving for the critical δ gives $\widehat{\delta} = \frac{9}{11}$.. Since this is bigger than $\frac{9}{17}$, this is the critical value for these strategies to be an SPE.
     - Note that the nodes from the previous bullet aren't very important – it would be easy to adjust the strategies so that this would be an SPE for $\widehat{\delta} = \frac{9}{17}$. This will be my "trick" for part (d).

   (c)
   - Yes. There are many possibilities. Here's one:

- Normal mode: always exit and produce $q_I = \frac{1}{2}$. If enter, produce $q_I = 1$ and $q_E = 0$.
- Trigger mode: play the SPE from (a) forever.
- Stay in Normal mode unless Enter AND $q_I \neq 1$.

- Using single deviation principle, note that the only place where anyone would potentially want to deviate is the incumbent producing $q_I = 1$ after an entry. She is willing to do this whenever

$$0 + \frac{\delta}{1-\delta}\left(\frac{1}{4}\right) \geq \frac{1}{4} + \frac{\delta}{1-\delta}\left(\frac{1}{9}\right),$$

or when $\delta > \frac{9}{5}$. Since $.9 > \frac{9}{5}$,, we're good.

(d)

- Something like "carrot and stick" from class would work. But here it's easier if you got (b) right.
- Simply change the strategies from (b) from
  - if any producer has ever produced a quantity other than $\frac{1}{4}$, "trigger" to playing the subgame perfect equilibrium from part (a) in every period.
    TO
  - if any producer has ever produced a quantity other than $\frac{1}{4}$ AFTER ENTRY, "trigger" to playing the subgame perfect equilibrium from part (a) in every period.
    AND from
  - Every producer produces $q = \frac{1}{4}$, so long as no producer has ever produced a quantity other than $\frac{1}{4}$.
    TO
  - After entry, every producer produces $q = \frac{1}{4}$, so long as no producer has ever produced a quantity other than $\frac{1}{4}$ IN A ROUND WHERE THERE WAS ENTRY. After Exit, produce $q = \frac{1}{2}$.
- Then the same reasoning from (b) applies, and we get an equilibrium whenever $\widehat{\delta} = \frac{9}{17}$ or greater, which is lower than the correct answer from (b).