Q1. Third order, direct space time method. i) Derive a third order accurate (time and space) finite difference approximation to the linear advection problem

$$\partial_t \theta + c \partial_x \theta = 0 \quad (1)$$

where $c > 0$ a positive constant flow. The resulting scheme should take the form

$$\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) = -\frac{c}{\Delta x}(\delta \theta_{i-2}^n + \gamma \theta_{i-1}^n + \beta \theta_i^n + \alpha \theta_{i+1}^n) \quad (2)$$

where α, β, γ and δ are factors that you will determine. Assume a regular grid with index i such that $x_i = i\Delta x$ and $\theta_i = \theta(x_i)$. Hint: You will need higher time derivatives of the above governing equation to eliminate the first and second order time truncation terms.
ii) Derive the discrete flux F that when used in the difference equation

$$\frac{1}{\Delta t}(\theta_i^{n+1} - \theta_i^n) = -\frac{1}{\Delta x}(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}) \quad (3)$$

makes it equivalent to the difference equation (1). Hint: F takes the form

$$F_{i+\frac{1}{2}} = c[\theta_i + d_1(\theta_i - \theta_{i-1}) + d_0(\theta_{i+1} - \theta_i)] \quad (4)$$

where d_0 and d_1 are functions of the Courant number, $C = \frac{c\Delta t}{\Delta x}$.
iii) Consider this flux in the limit of vanishing Courant number. What discretization does this correspond to (see your previous problem set)?

Q2. Finite volume method Again, consider the linear advection problem cast in flux form (3) where $F = c\theta$ with $c > 0$ on a regular grid. We will consider the flux of properties across the point $x = x_{i+\frac{1}{2}}$ as the average of the upstream time-average of

$$F_{i+\frac{1}{2}} = \frac{1}{\Delta t}\int_{x_{i+\frac{1}{2}} - c\Delta t}^{x_{i+\frac{1}{2}}} \theta(x)\, dx \quad (5)$$

i) Consider the distribution of θ at time $t = n\Delta t$ assuming that θ is piecewise constant in the finite volume Δx around each point x_i (i.e. θ is constant with value θ_i between $x_i - \frac{1}{2}\Delta x$ and $x_i + \frac{1}{2}\Delta x$.).
a) Evaluate $F_{i+\frac{1}{2}}$ in equation (5). You may assume that $\Delta t \leq \Delta x/c$.
b) What is this scheme usually called?
c) To make this calculation, why is it useful to assume $\Delta t \leq \Delta x/c$?

d) Now re-evaluate $F_{i+\frac{1}{2}}$ in equation (5), this time assuming $\Delta x/c \leq \Delta t \leq 2\Delta x/c$.
e) Generalize you answers for (a) and (d) so that you can evaluate $F_{i+\frac{1}{2}}$ using one expression assuming $\Delta t \leq 2\Delta x/c$. Hint: you will need to use the min and max functions:

$$min(a,b) = \begin{cases} a & \text{if} \quad a \leq b \\ b & \text{if} \quad a > b \end{cases}$$

$$max(a,b) = \begin{cases} a & \text{if} \quad a \geq b \\ b & \text{if} \quad a < b \end{cases}$$

ii) Consider the distribution of θ at time $t = n\Delta t$ to be piecewise linear between the nodes x_i.
a) Write down θ as a function of x in the interval $x_i \leq x \leq x_{i+1}$. Hint: this is simply linear interpolation between the values θ_i and θ_{i+1}.
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise linear distribution. You may assume that $\Delta t \leq \frac{1}{2}\Delta x/c$.
c) What is this scheme usually called?
iii) Consider the distribution of θ at time $t = n\Delta t$ to be piecewise quadratic between the nodes x_i.
a) Write down θ as a function of x in the interval $x_i \leq x \leq x_{i+1}$ by fitting a quadratic function to the nodes θ_{i-1}, θ_i and θ_{i+1} (i.e $\theta(x_j) = \theta_j$ at $j = i-1, i, i+1$).
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) assuming a piecewise quadratic distribution.
c) In the limit of vanishing time-step, what scheme does the flux in (b) approach?
iv) Again, consider the distribution of θ at time $t = n\Delta t$ to be piecewise quadratic in the interval $x_i \leq x \leq x_{i+1}$ and to take the form:

$$\theta(x) = \alpha + 2\beta\frac{(x - x_{i+\frac{1}{2}})}{\Delta x} + 3\gamma\frac{(x - x_{i+\frac{1}{2}})^2}{\Delta x^2}. \tag{6}$$

a) Find α, β and γ so that the spatial average over each finite volume (Δx) around x_{i-1}, x_i and x_{i+1} equals θ_{i-1}, θ_i and θ_{i+1} respectively. Note that this is different to fitting the quadratic function at the nodes as you did in part (iii).
b) Evaluate $F_{i+\frac{1}{2}}$ in equation (5) using the "finite volume" representation from (a).
c) What is this scheme usually called?

Q3. Discrete conservation of variance The average and difference operators are

$$\overline{\theta}^i = \frac{1}{2}\left(\theta_{i+\frac{1}{2}} + \theta_{i-\frac{1}{2}}\right)$$
$$\delta_i\theta = \theta_{i+\frac{1}{2}} - \theta_{i-\frac{1}{2}}$$

a) Prove the discrete product rule

$$\delta_i(\overline{\theta}^i U) = \overline{U\delta_i\theta}^i + \theta\delta_i U.$$

b) Prove the discrete product rule

$$\delta_i(\theta\phi) = \overline{\theta}^i\delta_i\phi + \overline{\phi}^i\delta_i\theta.$$

c) A scalar advection equation and continuity equation are discretized

$$\Delta x\Delta y\partial_t\theta + \delta_i(\overline{\theta}^i U\Delta y) + \delta_j(\overline{\theta}^j V\Delta x) = 0$$
$$\delta_i(U\Delta y) + \delta_j(V\Delta x) = 0.$$

Prove that the global integral of variance ($\int\int \theta^2\,dx\,dy$) is conserved given no normal flow at domain boundaries. Assume perfect treatment of the time derivative.

Q4. Burgers equation (Matlab) Burgers equation is

$$\partial_t u + u\partial_x u = 0.$$

We will consider this equation in the re-entrant (periodic) domain $0 \leq x \leq 1$ (i.e. $u(x = 1, t) = u(x = 0, t)$ for all t).
i) Show that the continuous Burgers equation (globally) conserves $\int u^p\,dx$ where p is an integer.
ii) a) Spatially discretize Burgers equation using centered second order difference but keeping a continuous time derivative. This is known as a differential-difference equation.
b) Show that although the differential-difference equation (ii.a) was not written as the divergence of a flux, that this form does conserve $< u >$ (volume mean of u) and that it can be equivilently written in the flux form

$$\partial_t u = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)$$

where $F_{i+\frac{1}{2}}$ takes a particular form.

c) Show that the differential-difference equation (ii.a) does not conserve $< u^2 >$. You should arrive at the result

$$\sum_i \frac{1}{2}\partial_t u_i^2 = \sum_i \frac{1}{2\Delta x} u_i u_{i+1}(u_{i+1} - u_i)$$

d) Time discretize the differential-difference equation using the forward method.

Use the energy method to derive the numerical stability criteria of the for this discretization. The result takes the form

$$(1 - C_i^*)^2 \leq 1$$

where $C_i^* = \frac{\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$ is a proxy Courant number.

e) Write a Matlab script to solve the discrete Burger's equation (ii.d) using an initial condition of $u(x, t = 0) = \sin(2\pi x)$, $\Delta x = 1/50$ and $\Delta t = 1/1000$. Plot the solution, $u(x)$, at the two times $t = 0.15$ and $t = 0.2$. Plot the evolution of $< u^2 >$ for the interval $t = 0. \ldots 0.2$

iii) Burgers equation can be written in flux form as

$$\partial_t u + \frac{1}{2}\partial_x u^2 = 0.$$

and a corresponding flux-form differential-difference equation is

$$\partial_t u = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) \quad \text{with} \quad F_{i+\frac{1}{2}} = \frac{1}{4}\left((u_i)^2 + (u_{i+1})^2\right)$$

a) Show that the differential-difference equation (iii) does not conserve $< u^2 >$. You should arrive at the result

$$\sum_i \frac{1}{2}\partial_t u_i^2 = -\sum_i \frac{1}{4} u_i u_{i+1}(u_{i+1} - u_i)$$

b) Using the forward method, solve the discrete model (form iii) in Matlab and plot the solution as before at $t = .15$, $t = 0.2$ and the evolution of $< u^2 >$.

c) Noting the difference in the answers to (ii.c) and (iii.a), combine the two flux forms, (ii.b) and (iii), so that the corresponding differential-difference equations conserves both $< u >$ and $< u^2 >$.

d) Implement this form (iii.c) in your Matlab script and plot the solution and evolution of $< u^2 >$ as before. Why is $< u^2 >$ not constant?