Generation of internal gravity waves by flow over topography

Boussinesq equations

From $\rho = \rho_0(1 - b/g)$, $p = -\rho_0 g z + \rho_0 P$, and $b << g$ (and $c_s >> \sqrt{gH}$)

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla P + \hat{\mathbf{z}} b$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{D}{Dt} b = 0$$

Linearized equations

One mechanism for creating internal gravity wave is flow over topography. We'll consider the simple case with zonal flow U at a sinusoidal topography at $z = h_0 \cos(kx)$. The basic state is composed of a mean zonal flow $\overline{u} = U(z)$, a vertical stratification $\frac{\partial}{\partial z}\overline{b} = N^2(z)$ in hydrostatic balance, $\overline{P} = \int^z \overline{b}$. All deviations are negligible compared with the basic state. For simplicity we consider two dimensional problems only ($\frac{\partial}{\partial y} = 0$ for all perturbations).

The perturbation equations are,

$$
\begin{aligned}
\frac{\partial}{\partial t}u + U\frac{\partial}{\partial x}u + wU_z - fv &= -\frac{\partial}{\partial x}P \\
\frac{\partial}{\partial t}v + fu &= -\frac{\partial}{\partial y}P \\
\frac{\partial}{\partial t}w + U\frac{\partial}{\partial x}w &= -\frac{\partial}{\partial z}P + b \\
\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w &= 0 \\
\frac{\partial}{\partial t}b + U\frac{\partial}{\partial x}b + wN^2 &= 0
\end{aligned}
$$

Bottom boundary conditions

The condition at the bottom is of no normal flow.

$$(U\hat{\mathbf{x}} + \mathbf{u}) \cdot \hat{\mathbf{n}} = (U\hat{\mathbf{x}} + \mathbf{u}) \cdot \frac{\hat{\mathbf{z}} - \nabla h}{\sqrt{1 + |\nabla h|^2}} = 0$$

or

$$(U + u)\frac{\partial}{\partial x}h = w \quad at \quad z = h(x, y)$$

(We can find the normal by thinking about a function $F(x, y, z) = z - h(x, y)$; its three-dimensional gradient is perpendicular to the surfaces of constant F, in particular the one at $F = 0$ which represents the boundary.) This linearizes to

$$w = U\frac{\partial}{\partial x}h \quad at \quad z = 0$$

when the slope and the net height change is small.

Generation of lee waves with no rotation

The motions is two-dimensional and non-divergent and we can therefore write the linear problem in terms of the wave streamfunction,

$$u = -\frac{\partial}{\partial z}\psi, \qquad w = \frac{\partial}{\partial x}\psi.$$

The streamfunction satisfied the linear problem,

$$U^2 \frac{\partial^2}{\partial x^2}\left[\nabla^2\psi + \left(\frac{N^2}{U^2} - \frac{U_{zz}}{U}\right)\psi\right] = 0.$$

For motion that is periodic in x, we can integrate the above equation twice to obtain,

$$\nabla^2\psi + \left(\frac{N^2}{U^2} - \frac{U_{zz}}{U}\right)\psi = 0.$$

For solutions that are periodic in x with wavenumber k, we can write the streamfunction as,

$$\psi = \phi(z)e^{\imath kx}$$

where it is understood that we take the real part of the solution.

Thus ϕ satisfies,

$$\frac{\partial^2}{\partial z^2}\phi + \left(m^2(z) - k^2\right)\phi = 0, \qquad m^2 \equiv \frac{N^2}{U^2} - \frac{U_{zz}}{U}.$$

Boundary conditions

The boundary condition over a bumpy lower boundary in terms of a streamfunction is given by,

$$\psi(x,0,t) = Uh(x,y).$$

For a bumpy lower boundary with elevation given by

$$h = h_0 \cos(kx)$$

we have

$$\psi(x,0) = Uh_0 e^{\imath kx}, \qquad \phi(0) = Uh_0.$$

We will imagine that the upper boundary condition is very far and idealize that by considering that z runs between 0 at the lower boundary and infinity for large positive z.

Short scales

For a mean flow with zero shear, we have

$$\sqrt{k^2 + m^2} = \frac{N}{U} \quad or \quad m^2 = \frac{N^2}{U^2} - k^2$$

If the topographic scale is short compared to U/N, the m^2 will be negative so that if $\hat{m} = \sqrt{k^2 - N^2/U^2}$ then

$$\psi = U h_0 \, \Re(e^{\imath kx \mp \hat{m} z})$$

We must choose the negative sign so that the disturbance decays with height

$$\psi = U h_0 \cos(kx) \exp(-\sqrt{k^2 - N^2/U^2}\; z)$$

Long scales

If $k^2 < N^2/U^2$ then m is real and our solution looks like

$$\psi = Uh_0 \, \Re(e^{\imath kx \pm \imath mz})$$

and we must decide which sign to use (or have some contribution from each). We shall discuss a number of ways of resolving the issue.

Group Velocity: Since the topography is the source of the waves, we would expect the vertical component of c_g to be positive. This means that if we suddenly add or eliminate the topography, the disturbance in the wave field would propagate upwards. Therefore

$$\frac{\partial}{\partial m}\left[Uk - \frac{Nk}{\sqrt{k^2+m^2}}\right] = \frac{Nkm}{(k^2+m^2)^{3/2}} > 0$$

The positive sign is the correct one, so that

$$\psi = Uh_0 \cos(kx + \sqrt{N^2/U^2 - k^2}\; z)$$

Energy Flux: For these 2-D motions, we can write the average (as in zonal average) vertical energy flux as

$$\overline{wP} = \overline{\frac{\partial \psi}{\partial x} P} = -\overline{\psi \frac{\partial P}{\partial x}}$$

and we expect it to be positive. Using the zonal momentum equation gives

$$\overline{wP} = -\overline{\psi \frac{\partial^2 \psi}{\partial t \partial z}} - U\overline{\psi \frac{\partial^2 \psi}{\partial x \partial z}} = -\overline{\psi \frac{\partial^2 \psi}{\partial t \partial z}} + U\overline{\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial z}}$$

For steady flow with $\psi = Uh_0 \cos(kx \pm mz)$, we have

$$\overline{wP} = \pm \frac{1}{2} U^3 h_0^2 km$$

again showing the plus sign to be the desired one.

Damping: Another approach is to add damping to the equations so that even the vertically wavy mode decays and reject any growing solution. We take

$$\frac{\partial}{\partial t}\mathbf{u} + U\frac{\partial}{\partial x}\mathbf{u} = -\nabla P + b\hat{\mathbf{z}} - \epsilon\mathbf{u}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\frac{\partial}{\partial t}b + U\frac{\partial}{\partial x}b + wN^2 = -\epsilon b$$

We now have

$$(\imath kU + \epsilon)^2 = -\frac{N^2k^2}{k^2+m^2} \quad \Rightarrow \quad m^2 = \frac{N^2}{U^2(1-\imath\epsilon/kU)^2} - k^2$$

The imaginary part of m is

$$\Im(m) \simeq \frac{1}{\Re(m)}\frac{\epsilon N^2}{kU^3}$$

so that vertically decaying solutions $\Im(m) > 0$ require $\Re(m) > 0$ as before.

Initial Value Problem: Finally, we can look at what happens if we suddenly turn the flow or the topography on. Using

$$\left(\frac{\partial}{\partial t} + \imath kU\right)^2 \left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi = k^2 N^2 \psi$$

with the initial and boundary conditions

$$\psi(z,0) = 0 \quad , \quad \psi(0,t) = Uh_0 \quad , \quad \psi(\infty,t) = 0$$

The Laplace transformed problem gives the same z structure equation as in the damped system

$$\left(\frac{\partial^2}{\partial z^2} - k^2\right)\psi^T = -\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)}\psi^T$$

with

$$\psi^T(0,s) = Uh_0/s \quad , \quad \psi^T(\infty,s) = 0$$

Again the positive root is the proper one

$$\psi^T = Uh_0 \frac{1}{s}\exp(\imath\,[\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)} - k^2]^{1/2} z)$$

The inverse transform

$$\psi = Uh_0 \int_{-\imath\infty}^{\imath\infty} \frac{1}{s}\exp(\imath\,[\frac{N^2}{(U^2 - 2\imath sU/k - s^2/k^2)} - k^2]^{1/2} z) e^{st}$$

is dominated by the singularity at $s = 0$; for large time, we recover the standing wave solution.

The Nonlinear Problem

We can also look at the nonlinear problem in simple 2-D cases. The steady equations

$$\mathbf{u} \cdot \nabla q = -\frac{\partial}{\partial x} b$$
$$\mathbf{u} \cdot \nabla (b + N^2 z) = 0$$

can be solved by noting that $\mathbf{u} \cdot \nabla \phi = 0$ implies $\phi = \Phi(\psi)$ – the advected property is constant along streamlines, since the parcels of fluid move along the streamlines in steady flows. The streamfunction here includes both the mean flow and the fluctuations $\psi = Uz + \psi'(\mathbf{x})$. Therefore

$$N^2 z + b'(\mathbf{x}) = B(Uz + \psi'(\mathbf{x})) = \frac{N^2}{U}(Uz + \psi')$$

Uniqueness could be a problem, of course. In any case, we'll take

$$b' = \frac{N^2}{U}\psi'$$

The vorticity equation then tells us that

$$\mathbf{u} \cdot \nabla q = w \frac{N^2}{U} \quad \Rightarrow \quad \mathbf{u} \cdot \nabla (q - \frac{N^2}{U} z) = 0$$

so that

$$\nabla^2 \psi' - \frac{N^2}{U} z = Q(Uz + \psi') = -\frac{N^2}{U^2}(Uz + \psi')$$

or

$$\nabla^2 \psi' = -\frac{N^2}{U^2} \psi'$$

with the boundary conditions

$$\psi'(x, h) = Uh \quad , \quad \text{radiation conditionatinfinity}$$

Solutions can be found in the form,

$$\psi(x, h) = Uh(x) \cos(N(z - h)/U) + Uf(x) \sin(N(z - h)/U).$$

the function $f(x)$ is determined imposing the radiation condition.

Convective instability

The buoyancy field is given by,

$$b = N^2 z - \frac{N^2}{U}\psi.$$

The condition for convective instability is,

$$\frac{\partial}{\partial z} b = N^2 - \frac{N^2}{U}\frac{\partial}{\partial z}\psi < 0,$$

or

$$\frac{\partial}{\partial z}\psi > U.$$

For solutions of the form

$$\psi = U h_0 \, \Re(e^{\imath kx \pm \imath mz}), \qquad m^2 = \frac{N^2}{U^2} - k^2 \approx \frac{N^2}{U^2}$$

this implies

$$\frac{N h_0}{U} > 1.$$

Shear instability

The condition for shear instabilities is

$$Ri \equiv \frac{N^2 + b_z}{(U_z + u_z)^2} < \frac{1}{4}.$$

Nappo, Atmospheric gravity waves, 141-144.

Critical levels

Critical levels appear when the phase speed of the waves matches the mean flow speed. For stationary non-rotating lee waves this happens when $U = 0$. The vertical wavenumber $m \approx N/U \to \infty$, The positive sign is the correct one, so that

$$\psi = U h_0 \cos(kx + \sqrt{N^2/U^2 - k^2}\; z).$$

Group Velocity:

$$c_g \frac{\partial}{\partial m}\left[Uk - \frac{Nk}{\sqrt{k^2+m^2}}\right] = -\frac{Nkm}{(k^2+m^2)^{3/2}} \to 0$$

Energy Flux: The energy flux vanishes at the critical level.

$$\overline{wP} = \pm\frac{1}{2}U^3 h_0^2 km.$$

Nappo, Atmospheric gravity waves, 114-123.

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LeBlond and Mysak, Waves in the Ocean, 377-379.