Baroclinic Turbulence

Two layer model

We have the two PV equations

$$\frac{\partial}{\partial t}q' + U\frac{\partial}{\partial x}q' + Q_y v' + \frac{\partial}{\partial x}(u'q') + \frac{\partial}{\partial y}(v'q' - \overline{v'q'}) = -\mu q' - r\zeta'$$

$$q' = \nabla^2\psi' \mp F(\psi_1' - \psi_2')$$

Here $F = f_0^2/g'H$ and $Q_y = \beta - U_{yy} \pm F(U_1 - U_2)$.

The zonal average equations are

$$\frac{\partial}{\partial t}U = \overline{v'q'} + \mathcal{F} - rU - \mu U \pm F\phi \equiv R - \mu U \pm F\phi$$

using

$$\overline{w^\dagger} \equiv \frac{f_0}{g'}\frac{\partial}{\partial y}\phi \quad , \quad f_0\overline{v^\dagger} = \pm F\phi$$

The omega equation becomes

$$\left[\frac{\partial^2}{\partial y^2} - F_1 - F_2\right]\phi = \frac{g'}{f_0}\mathcal{H}_y + R_1 - R_2$$

with $R = \overline{v'q'} + \mathcal{F} - rU$.

Stability

We shall arrange the forcing to produce a uniform vertical shear so that $U_j \to \mathcal{U}_j + U_j$. The background PV gradients are

$$\mathcal{Q}_y = \beta \pm F(\mathcal{U}_1 - \mathcal{U}_2)$$

The Rayleigh/ Charney-Stern theorem gives a necessary condition for instability: the gradient Q_y must change sign either in the horizontal or in the vertical; this will occur when

$$\mathcal{U}_1 - \mathcal{U}_2 < -\beta_1/F_1 \quad or \quad \mathcal{U}_1 - \mathcal{U}_2 > \beta_2/F_2$$

Note the asymmetry between eastward vs. westward shear in the ocean where we typically take $F_2 \sim \frac{1}{5}F_1$: we require much more eastward shear to destabilize the flow.

To solve, we take $q' = \hat{q}\exp(i\mathbf{k}\cdot\mathbf{x} + \sigma t)$ with ψ defined similarly. from the PV specification, we have

$$\hat{q}_i = L_{ij}\hat{\psi}_j \quad or \quad \hat{\psi}_i = L_{ij}^{-1}\hat{q}_j$$

The dynamical equation becomes

$$(\sigma+\mu)\hat{q}_i = -\imath k\mathcal{U}_i\hat{q}_i - \imath k\mathcal{Q}_{y_i}\hat{\psi}_i + r|\mathbf{k}|^2\hat{\psi}_i$$

This is a standard eigenvalue problem

$$\left[-\imath k\mathbf{U} - \imath k\mathbf{Q}_y\mathrm{L}^{-1} + |\mathbf{k}|^2\mathbf{R}\mathrm{L}^{-1}\right]\mathbf{q} = (\sigma+\mu)\mathbf{q}$$

except the matrix is complex. The $\mathbf{U}$, $\mathbf{Q}_y$, and R matrices are diagonal (with the latter having $R_{11}=0$, $R_{22}=r$). For the two-layer model,

$$\mathrm{L} = \begin{pmatrix} -K^2 - F_1 & F_1 \\ F_2 & -K^2 - F_2 \end{pmatrix}$$

and

$$\mathrm{L}^{-1} = -\frac{1}{K^2(K^2+F_1+F_2)}\begin{pmatrix} K^2+F_2 & F_1 \\ F_2 & K^2+F_1 \end{pmatrix}$$

Demos, Page 2: Growth rates <r=0,u1 pos F2=4> <r=0,u1 neg F1=20> <r=0.01,u1 pos> <r=0.01,u1 neg> <r=0.1,u1 pos> <r=0.1,u1 neg>

Other problems

The Eady model just has a different L: if subscript 1 is the top and 2 is the bottom, then

$$\psi = \psi_1\frac{\sinh Kz}{\sinh KH} + \psi_2\frac{\sinh K(H-z)}{\sinh KH}$$

with $K=\sqrt{k^2+\ell^2}N/f$. The active scalars are $q_1 = \frac{\partial}{\partial z}\psi|_H$ and $q_2 = \frac{\partial}{\partial z}\psi|_0$ and

$$\mathrm{L} = \begin{pmatrix} K\coth KH & -K\,\mathrm{csch}KH \\ K\,\mathrm{csch}KH & -K\coth KH \end{pmatrix}$$

Demos, Page 2: eady <sigma vs r>

Cessation of growth

The linear problem predicts exponential growth even with friction; to stop this, the nonlinearity must enter at some point. This could occur in the form of divergent eddy PV fluxes

$$\frac{\partial}{\partial t}Q = -\frac{\partial}{\partial y}\overline{v'q'} + \ldots$$

which alter the mean PV and therefore the mean U fields. In turn, the linear part of the fluctuation eqn. changes and the growth rate can drop.

The mean field approximation discussed previously relies on this mechanism. Because of channel walls, the growing modes have

$$q_i' = \hat{q}_i(t)\cos(kx-\theta_i)\sin(\ell y)$$

and the phases differ between the two q's and therefore between the ψ's and q's. In that case, the v' will not be in quadrature with the q'. The linear solution predicts a PV flux proportional to $\sin(2\ell y)$ which tries to eliminate the negative Q_y gradient in the center of the channel.

Demos, Page 2: Examples <F=100,beta=0.1> <full> <mean>

Residual circulation

We can compare the residual circulation, represented by ϕ, to the Eulerian mean meridional flow, which we can represent by

$$\overline{v^a} = \overline{v^\dagger} - \overline{v'h'}$$

or

$$\phi^a = \phi - \overline{v'(\psi_1' - \psi_2')}$$

These are for $F = 100$, $\beta = 0.1$.

`Demos, Page 3: Diagnostics <Eddy enstrophy> <Zonal flow> <mean> <PV gradients> <mean PV gradients> <phi> <phia> <mean>`

Infinite/ doubly-periodic domain

In this case the divergence of the eddy fluxes vanishes, so there is no feedback on the mean. The linearly growing wave is an exact solution to the fully nonlinear equations: if

$$\mathbf{q}' = f(kx + \ell y) = f(\xi)$$

then the inversion implies $\psi' = g(\xi)$ with

$$(k^2 + \ell^2)g \mp F(g_1 - g_2) = f$$

But the nonlinear term is now

$$J(\psi', q') = k\ell f'g' - \ell k f'g' = 0$$

Pedlosky (1975) and Pedlosky and Polvani (1987) suggest that the growing wave can itself become unstable to other waves which then remove the energy. We can illustrate this by using a wave triad in which the middle wavenumber is the baroclinically unstable one while the smaller and larger ones are stable.

$$\frac{\partial}{\partial t} q_1 = -\imath k \mathcal{U} q_1 - \imath k \mathcal{Q}_y \psi_1 + (k_2\ell_3 - k_3\ell_2)(\psi_3^* q_2^* - \psi_2^* q_3^*) + r K_1^2 \psi_1$$

The indices are triad number; these need to be solved in each layer so that there are 12 degrees of freedom (6 complex q's).

`Demos, Page 3: triad <no nonlinearity> <nonlinear> <full calculation> <full enstrophies> <full enstrophies> <full energy> <zonal flow>`

The full solution shows this breakdown and the development of turbulence. With these parameters, there are zonal means with amplitudes (at the end) of about 1/3 the maximum zonal velocity.

Upper bounds

Shepherd (1988, JAS) develops an upper bound for the wave enstrophy based on the nonlinear stability theory of Arnol'd. It's worthwhile starting from that point since it can give insight into nonzonal systems as well.

Liapunov stability

Essentially we want to derive a functional which has vanishing derivative at the basic state and then determine if it has a minimum, maximum, or saddle point there. In ther first two cases, the basic state will be stable. For the linearized problem such a functional can be derived directly and is proportional to the square of the perturbation amplitude (in the unstable case having indeterminate sign for the coefficients at various wavenumbers).

We start with the PV equation and split it into basic state and fluctuation

$$J(\Psi, Q) = -\mu Q + \mu Q_f$$

$$\frac{\partial}{\partial t} q + J(\Psi, q) + J(\psi, Q) + J(\psi, q) = -\mu q$$

We presume we can take $Q = Q_f$ and that the corresponding Ψ satisfies

$$J(\Psi, Q) = 0$$

For the channel with Q_f depending on y, this is straightforward. This relationship implies

$$Q = \mathcal{Q}(\Psi) \quad or \quad \Psi = \mathcal{P}(Q)$$

Then we can write the linearized stability problem as

$$\frac{\partial}{\partial t} q' + J(\Psi, q' - \mathcal{Q}_\Psi \psi') = 0$$

The perturbation enstrophy satisfies

$$\frac{\partial}{\partial t} \frac{q'^2}{2} + J(\Psi, \frac{q'^2}{2}) - \mathcal{Q}_\Psi q' J(\Psi, \psi') = -\mu q'^2$$

while the total energy obeys

$$\frac{\partial}{\partial t} E = \int \psi' J(\Psi, q') - 2\mu E = -\int q' J(\Psi, \psi') - 2\mu E$$

The first equation implies

$$\frac{\partial}{\partial t} \int \frac{q'^2}{2\mathcal{Q}_\Psi} - \int q' J(\Psi, \psi') = -2\mu \int \frac{q'^2}{2\mathcal{Q}_\Psi}$$

So that the Arnol'd invariant simply decays with time.

$$\frac{\partial}{\partial t}\mathcal{A} = -2\mu\mathcal{A} \quad , \quad \mathcal{A} = E + \int \frac{q'^2}{2\mathcal{Q}_\Psi}$$

We thus obtain the two theorems:

- Unstable flows imply that the energy and enstrophy are growing in time; this can only happen if the second term can offset the first. Thus, if $\mathcal{Q}_\Psi > 0$ everywhere, the flow is stable.
- If $\mathcal{Q}_\Psi$ is everywhere negative, the flow can still be unstable if the available wavenumbers are so large that $\mathcal{A}$ is negative definite.

For nonlinear stability, we define the Hamiltonian functional

$$H[q] = -\int\int q(\mathbf{x})G(\mathbf{x}|\mathbf{x}')q(\mathbf{x}')$$

and a so-called Casimir

$$C[q] = \int\int^q \mathcal{P}(s)ds$$

under the assumption that $\mathcal{P}$ is monotonically increasing (and ignoring the boundary terms – see Shepherd, *Adv. in Geophys,* **32**, 287-338 for a more formal derivation). Then

$$\begin{aligned}\mathcal{E}[q] &= H[q] - H[Q] + C[q] - C[Q] \\ &= \int \frac{1}{2}|\nabla\psi'|^2 + \frac{1}{2}\frac{f^2}{N^2}|\frac{\partial\psi'}{\partial z}|^2 + \int_0^{q'}[\mathcal{P}(Q+s) - \mathcal{P}(Q)]ds\end{aligned}$$

is conserved, is equal to zero when $q' = 0$ and its first variation also vanishes (it's quadratic in q'). $\mathcal{E}$ is called the "pseudoenergy." If, therefore,

$$0 < C_{min} < \mathcal{P}_\Psi < C_{max}$$

and

$$||\psi'||^2 = E + C_{min}Z$$

then

$$||\psi'(t)||^2 \leq \frac{C_{max}}{C_{min}}||\psi'(0)||^2$$

– the flow is nonlinearly stable.

For flows with translational invariance, the pseudomomentum

$$\mathcal{M} = \int yq' - \int_Q^{Q+q'} Y(s)ds = -\int\int_0^{q'}[Y(Q+s) - Y(Q)]ds$$

is also conserved. $Y(Q)$ is the inverse of $Q(y)$. The stability theorem now states that

$$0 < C_{min} < |Q_y| < C_{max} \quad \Rightarrow \quad \int q'^2(t) \leq \frac{C_{max}}{C_{min}}\int q'^2(0)$$

For small amplitude, the psuedomomentum becomes the negative of the wave activity

$$\mathcal{M} = -\int q'^2 Y'(Q) = -\int \frac{q'^2}{Q_y}$$

Bounds

Shepherd considers the flow to be a **stable** zonal shear plus a deviation which includes a zonal shear which shifts the total into the unstable range plus the wave disturbance

$$q \to \mathcal{Q} + q = \mathcal{Q} + Q + q'$$

This derivation follows Shepherd (and Held's suggestion) dirctly except the layer depths are not presumed to be equal. For the Phillip's problem, we take $\mathcal{U}$ to be uniform. Then

$$\frac{\partial}{\partial t}\frac{1}{2}\int q_j^2 + \mathcal{Q}_j' \int v_j q_j = -\mu q_j^2$$

and the conservation follows directly

$$\frac{\partial}{\partial t}\mathcal{M} = \frac{\partial}{\partial t}\left[\frac{H_1}{2\mathcal{Q}_1'}\int q_1^2 + \frac{H_2}{2\mathcal{Q}_2'}\int q_2^2\right] = -2\mu\mathcal{M}$$

For $\mathcal{Q}_1' > \mathcal{Q}_2' > 0$, we have

$$\frac{1}{2}\int H_1 q_1^2 + H_2 q_2^2 \le \mathcal{Q}_1'\mathcal{M} \le \mathcal{Q}_1'\mathcal{M}(0) = \frac{1}{2}\int H_1 q_1^2(0) + H_2\frac{\mathcal{Q}_1'}{\mathcal{Q}_2'}q_2^2(0)$$

The PV gradients of the background flow are

$$\mathcal{Q}_1' = \beta + F_1\mathcal{U} \quad , \quad \mathcal{Q}_2' = \beta - F_2\mathcal{U}$$

and the PV's of the remainer are

$$q_1 = F_1(U-\mathcal{U})(y-\frac{1}{2}) + q_1' \quad , \quad q_2 = -F_2(U-\mathcal{U})(y-\frac{1}{2}) + q_2'$$

The right-hand side of the inequality (assuming the waves start off at infinitesimal amplitude) becomes

$$\frac{1}{24}(U-\mathcal{U})^2\left[H_1F_1^2 + H_2F_2^2\frac{\beta+F_1\mathcal{U}}{\beta-F_2\mathcal{U}}\right]$$

$$= \frac{1}{24}(U-\mathcal{U})^2\left[\frac{\beta(H_1F_1^2+H_2F_2^2)}{\beta-F_2\mathcal{U}}\right]$$

$$= \frac{1}{24}\frac{\beta H_1F_1}{R_d^2}\frac{(U-\mathcal{U})^2}{\beta-F_2\mathcal{U}}$$

since $H_1F_1 = H_2F_2$. We want to minimize this subject to

$$-\frac{\beta}{F_1} < \mathcal{U} < \frac{\beta}{F_2}$$

(Shepherd uses $\mathcal{U} > 0$ but that doesn't seem necessary).

We have either

$$\mathcal{U} = 2\frac{\beta}{F_2} - U \quad \text{(weak supercriticality)}$$

or

$$\mathcal{U} = -\frac{\beta}{F_1} \quad \text{if} \quad U > 2\frac{\beta}{F_2} + \frac{\beta}{F_1}$$

For the first case

$$\frac{1}{2}\frac{1}{H}\int H_1 q'^2_1 + H_2 q'^2_2 \leq \frac{\beta H_2}{6H_1}(UF_2 - \beta)$$

and for strong shear

$$\frac{1}{2}\frac{1}{H}\int H_1 q'^2_1 + H_2 q'^2_2 \leq \frac{H_1}{24H}(UF_1 + \beta)^2$$

For the run shown ($U = 1$, $F1 = F_2 = 100$, $\beta = 10$), this works out to about 250(?).

`Demos, Page 7: example <Eddy enstrophy>`