Chapter 7

Passive tracer spectra and 3D turbulence

For a passive scalar which obeys an equation of the form ,

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \kappa\nabla^2\theta, \tag{7.1}$$

we can write an equation for the variance $\langle\theta^2\rangle$,

$$\frac{\partial\langle\theta^2\rangle}{\partial t} + \nabla\cdot\langle\mathbf{u}\theta^2\rangle = -\kappa\langle|\nabla\theta|^2\rangle. \tag{7.2}$$

We assumed without loss of generality that $\langle\theta\rangle = 0$. Under the assumption that the tracer statistics are homogeneous and isotropic, we can write an equation for the spectrum $P(k)$ of this variance, analogous to (6.28),

$$2\kappa k^2 P(k) = T(k) + F(k), \tag{7.3}$$

where $T(k)$ is the nonlinear transfer of tracer variance, and $F(k)$ is an external source of tracer variance. Two of the results derived for the kinetic energy spectrum carry over to the tracer spectrum problem. (1) The dissipation of variance χ must equal the total injection of variance $\int_0^\infty F(k)\mathrm{d}k$. (2) At wavenumbers far from the injection scale and dissipation scale, variance is fluxed at a constant rate χ (set by the injection rate). Using these two results, we can derive the form of the spectrum $P(k)$. Notice however that there is a major difference between the kinetic energy and the tracer problems. In the tracer inertial range χ and k are not the only relevant parameters, since the tracer field is subject to stirring by the flow. The flow parameters (e.g., ϵ) also influence the tracer field.

We can derive the shape of the tracer spectrum in the range of wavenumbers where both tracer and momentum dissipation can be neglected. Once again we assume that forcing is confined to large scales. In the so-called *inertial-convective range* the fluxes of kinetic energy and tracer variance must be constant, if a statistically steady sate is to be achieved. Thus we can state, in analogy to Obukhov's argument for kinetic energy, that the tracer flux is given by the available variance at wavenumber k divided by the eddy turnover timescale,

$$\chi \sim \frac{kP(k)}{\tau}. \tag{7.4}$$

Assuming that eddy stirring is dominated by local interactions we can write that $\tau = [k^3 E(k)]^{-1/2}$. But χ is a constant and therefore we have,

$$P(k) \sim \chi k^{-5/2} E(k)^{-1/2} \tag{7.5}$$

Substituting for $E(K)$ from K41 we have,

$$P(k) = \beta \chi \epsilon^{-1/3} k^{-5/3} \tag{7.6}$$

where β is some universal constant. The tracer spectrum in the inertial-convective range has the same slope as the kinetic energy spectrum and is known as the Obukhov-Corrsin spectrum.

Length scales

The kinetic energy spectrum becomes influenced by viscosity at a wavenumber k_d such that $Re \sim 1$. In order to estimate the Reynolds number at a particular lengthscale, we need a scaling for the velocity field. Using K41 we have,

$$\langle \delta v_r^2 \rangle \sim (\epsilon r)^{2/3} \qquad \Longrightarrow \qquad v_r \sim (\epsilon r)^{1/3}, \tag{7.7}$$

where v_r is an order of magnitude estimate of the velocity at a lengthscale r. Then

$$Re_r \sim \frac{v_r r}{\nu}. \tag{7.8}$$

Setting $Re_r \sim 1$, we find that viscosity becomes important at the scale $1/r = k_d \sim (\epsilon/\nu^3)^{1/4}$, the Kolmogorov scale.

By analogy with the kinetic energy spectrum, the passive tracer spectrum becomes influenced by diffusion at a wavenumber k_c, where the Peclet number ~ 1. We have two different scenarios, depending on whether the wavenumber k_c is smaller or larger than the Kolmogorov wavenumber k_d.

If the Prandtl number $Pr = \nu/\kappa < 1$, then the dissipation scale k_c occurs within the inertial range ($k_c < k_d$). Plugging $v_r \sim (\epsilon r)^{1/3}$ in the definition of the Peclet number,

$$Pe_r \sim \frac{v_r r}{\kappa}, \tag{7.9}$$

we find that $Pe_r \sim 1$ is achieved at a wavenumber $1/r = k_c \sim (\epsilon/\kappa^3)^{1/4} = Pr^{3/4}k_d$.

However, if diffusion becomes important at wavenumbers larger than viscosity does (i.e. $Pr > 1$), k_c does not lie within the inertial range, so we cannot use the inertial range scaling to obtain v_r; if the energy spectrum $E(k)$ drops off more rapidly than k^{-3}, then $(\delta v_r)^2$ cannot be calculated from (6.50). In this range the velocity spectrum drops off exponentially to zero. Thus at scales k shorter than the Klolmogorov scale, the tracer is not stirred by eddies with scale k because such eddies do not exist. At these scales the trcaer is stirred by the smallest scales present in the flow, i.e. by eddies at the Kolmogorv scale. For these eddies $v_r \sim (\epsilon/k_d)^{1/3} \sim \nu k_d$. Smaller scale features feel this as a "large-scale" flow. Then the local Peclet number at a scale r is,

$$Pe_r = \frac{v_r r}{\kappa} = \frac{\nu k_d r}{\kappa}. \tag{7.10}$$

By definition $Pe_r \sim 1$ when $r = 1/k_c$, the wavenumber at which diffusion becomes important. Thus,

$$k_c \sim \frac{\nu}{\kappa} k_d. \tag{7.11}$$

Depending on the relative length of the viscous and dissipative cutoff scales, the passive tracer tracer spectrum has several different subranges. For $k_i << k$, and $k << k_d$ and $k << k_c$, neither κ nor ν are important. This is the *inertial-convective range* considered above. If $k << k_d$, but $k > k_c$ (for $Pr < 1$) then κ is important, but not ν: the spectrum is in an *inertial-diffusive range*. If $k << k_c$, but $k > k_d$ (for $Pr > 1$), then ν is important but not κ: the spectrum is in an *viscous-convective range*. Finally for $k > k_d$ and $k > k_c$, the spectrum is in a *viscous-diffusive range*. We consider the spectrum in each of these subranges separately.

Inertial-diffusive subrange

In the inertial diffusive range the flux of variance is no longer constant with k, since diffusion is acting to reduce it. Instead, from (7.3),

$$T(k) = -\frac{d\Pi}{dk} = 2\kappa k^2 P(k). \tag{7.12}$$

The flux $\Pi(k)$ is not a constant in k in this range. Using Obukhov's argument we can also write,

$$\Pi(k) = \frac{kP(k)}{[k^3 E(k)]^{-1/2}}. \tag{7.13}$$

Inertial range scaling for the energy still applies, so we can use K41 to express $E(k)$ and we find that,

$$P(k) \sim \Pi(k) k^{-5/2} E(k)^{-1/2} = \beta \epsilon^{-1/3} k^{-5/3} \Pi(k). \tag{7.14}$$

Substituting for $P(k)$ in (7.12) we have,

$$\frac{d\Pi}{dk} = -2\beta\kappa\epsilon^{-1/3}k^{1/3}\Pi(k). \tag{7.15}$$

Solving for $\Pi(k)$ we get,

$$\Pi(k) = \chi \exp\left[-\frac{3}{2}\beta\kappa\epsilon^{-1/3}k^{4/3}\right]. \tag{7.16}$$

If we substitute back into (7.14) we find,

$$P(k) = \beta\epsilon^{-1/3}k^{-5/3}\chi \exp\left[-\frac{3}{2}\beta\left(\frac{k}{k_c}\right)^{4/3}\right] \tag{7.17}$$

where $k_c = (\epsilon/\kappa^3)^{1/4}$. Hence the spectrum of tracer variance behaves exponentially for $k > k_d$ when $Pr < 1$. This spectrum is not valid far into the inertial-diffusive subrange because it assumes $\Pi(k)$ varies only slowly with k. (An alternative theory of Batchelor et al. (1959) gives a $k^{-17/3}$ spectrum. Neither form of the spectrum has been verified.)

Viscous-convective subrange

For $Pr > 1$ and $k > k_d$, but $k < k_c$, the flux of variance $\Pi(k)$ is constant: $\Pi(k) = \chi$. κ is not important, but ν is. The energy field drops off rapidly for $k > k_d$. Hence the scalar perturbations experience a shear corresponding to that at a scale k_d, $v_{kd}k_d = (\epsilon/\nu)^{1/2}$. At $k > k_d$ this shear appears like a smooth large-scale flow. $P(k)$ must satisfy,

$$\chi = \frac{kP(k)}{[k_d^3E(k_d)]^{-1/2}}. \tag{7.18}$$

Plugging the expression for the Kolmogorov wavenumber k_d,

$$P(k) = C_B\chi k^{-1}\left(\frac{\epsilon}{\nu}\right)^{-1/2}. \tag{7.19}$$

This is known as the **Batchelor spectrum**, and C_B is the Batchelor constant.

There is experimental evidence for the Batchelor spectrum. Gibson and Schwarz (JFM, 1963) observed the Batchelor spectrum for temperature and salinity in laboratory measurements in water, and the approximate behavior for temperature spectrum is also suggested by field measurements of Grant et al. (JFM, 1968), Oakey and Elliott (JPO, 1982) and others. There is however a wide scatter in the predicted values of the universal constant C_B. The practical importance of these spectral expressions lies in the fact that all scalar fluctuations and scalar dissipation are effectively determined by scales from the Batchelor range. The dissipation rates in turn determine the mixing coefficients for scalars which are critical to understand small-scale physics of the oceans and large scale circulation and global climate. The knowledge of spatial power spectra of temperature fluctuations at small scales is also needed in treating problems of sound and light propagation in water.

Further reading: Lesieur, Ch VI; Tennekes and Lumley, Ch 8.