Active Tracers

We review mixing length theory applied to a set of active scalars (think in terms of biological properties):

$$\frac{D}{Dt}b_i + \nabla \cdot (\mathbf{u}_{bio} b_i) - \nabla\kappa\nabla b_i = \mathcal{B}_i(\mathbf{b}, \mathbf{x}, t)$$

Split the field into an eddy part which varies rapidly in space and time and a mean part which changes over larger (order $1/\epsilon$) horizontal distances and longer (order $1/\epsilon^2$) times:

$$b_i = \overline{b}_i(z|\mathbf{X}, T) + b_i'(\mathbf{x}, z, t|\mathbf{X}, T)$$

We must allow for short vertical scales in both means and fluctuations. Counterbalancing this difficulty is the fact that vertical velocities tend to be weak (order $\frac{U}{fL} \times \frac{UH}{L}$). We assume the mean flows are small $\overline{u} \sim \varepsilon\mathbf{u}'$ and the coefficients in the reaction terms vary rapidly in the vertical but slowly horizontally and in time.

$$\frac{D}{Dt} - \nabla\cdot\kappa\nabla \longrightarrow$$
$$\left[\frac{\partial}{\partial t} + u_m'\nabla_m - \nabla\cdot\kappa\nabla - \frac{\partial}{\partial z}\kappa_v\frac{\partial}{\partial z}\right]$$
$$+\varepsilon\left[u_m' \;\boldsymbol{\nabla}_m + \overline{u}_m\nabla_m - \boldsymbol{\nabla}_m\kappa_{mn}\nabla_n - \nabla_m\kappa_{mn}\boldsymbol{\nabla}_n + \frac{Ro}{\varepsilon}w'\frac{\partial}{\partial z}\right]$$
$$+\,\varepsilon^2\left[\frac{\partial}{\partial T} + \overline{u}_m\boldsymbol{\nabla}_m - \boldsymbol{\nabla}_m\kappa_{mn}\boldsymbol{\nabla}_n + \frac{Ro}{\varepsilon}\overline{w}\frac{\partial}{\partial z}\right]$$

Vertical Structure

1) We assume the case with no flow has a *stable* solution:

$$\frac{\partial}{\partial z}w_{bio}\overline{b}_i = \frac{\partial}{\partial z}\kappa_v\frac{\partial}{\partial z}\overline{b}_i + \mathcal{B}_i(\overline{\mathbf{b}}, z|\mathbf{X}, T)$$

`Demos, Page 1:  bio dynamics` <*growth rates*>

2) The eddy-induced perturbations satisfy

$$\left[\frac{\partial}{\partial t} + \mathbf{u}'\cdot\nabla - \nabla\cdot\kappa\nabla\right]b_i' + \frac{\partial}{\partial z}w_{bio}b' =$$
$$\sum_j \frac{\partial\mathcal{B}_i}{\partial b_j}b_j' - \mathbf{u}'\cdot\boldsymbol{\nabla}\overline{b}_i \equiv \mathcal{B}_{ij}b_j' - \mathbf{u}'\cdot\boldsymbol{\nabla}\overline{b}_i$$

with $\boldsymbol{\nabla} = (\partial/\partial X,\ \partial/\partial Y,\ \partial/\partial z)$.

3) The equation for the mean is

$$\left[\frac{\partial}{\partial T} + \overline{\mathbf{u}}\cdot\boldsymbol{\nabla} - \boldsymbol{\nabla}\kappa\boldsymbol{\nabla}\right]\overline{b}_i + \boldsymbol{\nabla}\cdot(\overline{\mathbf{u}'b'}) + \frac{\partial}{\partial z}w_{bio}\overline{b}_i =$$
$$\mathcal{B}_i(\overline{\mathbf{b}}, z|\mathbf{X}, T) + \frac{1}{2}\frac{\partial^2\mathcal{B}_i}{\partial b_j\partial b_k}\overline{b_j'b_k'}$$

Summary:

Eddies generate fluctuations by horizontal and vertical advection of large-scale gradients, but the strength and structure depends on the biologically-induced perturbation decay rates.

Perturbations generate eddy fluxes and alter the average values of the nonlinear biological terms.

NPZ

A simple biological model (mixed layer):

$$\frac{D}{Dt}P = \frac{\mu PN}{N+k_s} - \frac{g}{\nu}Z[1-\exp(-\nu P)] - d_P P + \nabla\kappa\nabla P$$
$$\frac{D}{Dt}Z = \frac{ag}{\nu}Z[1-\exp(-\nu P)] - d_Z Z + \nabla\kappa\nabla Z$$
$$\frac{D}{Dt}N = -\frac{\mu PN}{N+k_s} + \frac{(1-a)g}{\nu}Z[1-\exp(-\nu P)]$$
$$+ d_P P + d_Z Z + \nabla\kappa\nabla N$$
$$or \quad N = N_T - P - Z$$

Mean-field approach

We can get a very similar picture using the mean-field approximation: take

$$\frac{\partial}{\partial t}\overline{b}_i + \overline{u}\cdot\nabla\overline{b}_i + \nabla\cdot(\overline{\mathbf{u}'b'}) + \frac{\partial}{\partial z}w_{bio}\overline{b}_i - \nabla\kappa\nabla\overline{b}_i = \overline{\mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t)}$$

$$\simeq \mathcal{B}_i(\overline{\mathbf{b}}, z|\mathbf{x},t) + \frac{1}{2}\frac{\partial^2\mathcal{B}_i}{\partial b_j \partial b_k}\overline{b'_j b'_k}$$

$$\frac{\partial}{\partial t}\mathbf{b}'_i + \overline{u}\cdot\nabla\mathbf{b}'_i + \nabla\cdot(\mathbf{u}'b'_i - \overline{\mathbf{u}'b'}) + \frac{\partial}{\partial z}w_{bio}\mathbf{b}'_i - \nabla\kappa\nabla\mathbf{b}'_i =$$
$$-\mathbf{u}'\cdot\nabla\overline{b}_i + \mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t) - \overline{\mathcal{B}_i(\overline{\mathbf{b}}+\mathbf{b}',\mathbf{x},t)}$$

or (dropping the quadratic and higher terms)

$$\frac{\partial}{\partial t}\mathbf{b}'_i + \overline{u}\cdot\nabla\mathbf{b}'_i + \frac{\partial}{\partial z}w_{bio}\mathbf{b}'_i - \nabla\kappa\nabla\mathbf{b}'_i \simeq -\mathbf{u}'\cdot\nabla\overline{b}_i + \mathcal{B}_{ij}b'_j$$

The differences are subtle: the MFA does not presume that the scale of $\overline{b}_i$ is large but linearizes in a way which may not be consistent.

Separable Problems

The mesoscale eddy field has horizontal velocities in the near-surface layer which are nearly independent of z, and the vertical velocity increases linearly with depth $w' = s(\mathbf{x}, t)z$. The stretching satisfies

$$s(\mathbf{x}, t) = -\nabla \cdot \mathbf{u}(\mathbf{x}, t)$$

For linear (or linearized perturbation) problems in the near-surface layers, we can separate the physics and the biology using Greens' functions.

We define the Greens function for the horizontal flow problem:

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla - \nabla \kappa \nabla\right) G(\mathbf{x}, \mathbf{x}', t - t') = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

The perturbation equations can now be solved:

$$\begin{aligned} b_i' = &- \int d\mathbf{x}' \int dt' G(\mathbf{x}, t|\mathbf{x}', t') u_m'(\mathbf{x}', t')\phi_{m,i}(z, t - t') \\ &- \int d\mathbf{x}' \int dt' G(\mathbf{x}, t|\mathbf{x}', t') s'(\mathbf{x}', t')\varphi_i(z, t - t') \end{aligned}$$

The two functions representing the biological dynamics both satisfy

$$\frac{\partial}{\partial \tau}\varphi_i = \frac{\partial}{\partial z}\kappa_v \frac{\partial}{\partial z}\varphi_i + \mathcal{B}_{ij}\varphi_j$$

with $\mathcal{B}_{ij} = \partial \mathcal{B}_i / \partial b_j$. These give the diffusive/ biological decay of standardized initial perturbations

$$\phi_{m,i}(z, 0) = \boldsymbol{\nabla}_m \bar{b}_i \quad , \qquad \varphi_i(z, 0) = z\frac{\partial}{\partial z}\bar{b}_i$$

`Demos, Page 3: perturbation structures` *<p'z' struct>* *<ev of p'>* *<ev of z'>*

Simple Example

If we ignore vertical diffusion and advection and consider only one component with $\mathcal{B}_{11} = -\lambda$, we have

$$\phi_{m,i} = e^{-\lambda\tau}\nabla_m \overline{b}_i$$

so that

$$b'_i = -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} G(\mathbf{x},t|\mathbf{x}',t') u'_n(\mathbf{x}',t')\right] \nabla_n \overline{b}_i$$

The eddy flux takes the form

$$\begin{aligned}\overline{u'_m b'} &= -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} \overline{u'_m(\mathbf{x},t) G(\mathbf{x},t|\mathbf{x}',t') u'_n(\mathbf{x}',t')}\right] \nabla_n \overline{b}_i \\ &= -\left[\int d\mathbf{x}' \int dt' e^{-\lambda(t-t')} R_{mn}(\mathbf{x},t|\mathbf{x}',t')\right] \nabla_n \overline{b}_i\end{aligned}$$

If we split the right-hand side into symmetric and antisymmetric parts, we find

$$\begin{aligned}\overline{u'_m b'} &= -K^\lambda_{mn} \nabla_n \overline{b}_i + \epsilon_{mnk} \Psi^\lambda_k \nabla_n \overline{b}_i \\ &= -K^\lambda_{mn} \nabla \overline{b}_i - (\epsilon_{mnk} \nabla_n \Psi^\lambda_k) \overline{b}_i + \epsilon_{mnk} \nabla_n (\Psi^\lambda_k \overline{b}_i)\end{aligned}$$

The last term has no divergence and can be dropped. Thus the eddy flux is a mix of diffusion and Stokes' drift:

$$\overline{u'_m b'} = -K^\lambda_{mn} \nabla \overline{b}_i + V^\lambda_m \overline{b}_i$$

Both coefficients depend on the biological time scale λ^{-1}.

For the random Rossby wave case, the Stokes drift term is

$$\mathbf{V}^\lambda = \frac{KE}{(\gamma+\lambda)^2 + \frac{1}{4}} \big(-\cos(2y) \ , \ \ 0 \big)$$

while the diffusivity tensor is

$$K^\lambda_{ij} = 2(\gamma+\lambda) \frac{KE}{(\gamma+\lambda)^2 + \frac{1}{4}} \begin{pmatrix} \cos^2(y) & 0 \\ 0 & \sin^2(y) \end{pmatrix}$$

`Demos, Page 4:  effective coeff`   <*effective k,v*>

Not so simple example

"Mixing length" models

$$Flux(b) = -\kappa_e \nabla b$$

even if appropriate for passive tracers are not suitable for biological properties whose time scales may be comparable to those in the physics. Instead, we find

$$\overline{\mathbf{u}'_m b'_i} = -\left[\int d\tau e^{\mathcal{B}_{ij}\tau} R_{mn}(\tau)\right] \nabla_n \overline{b}_j$$

where R_{mn} is the equivalent of Taylor's Lagrangian covariance (but including κ).

We divide the coefficient into symmetric (K) and antisymmetric terms related to the Stokes drift (V)

$$\overline{\mathbf{u}'_m b'_i} = -K^{ij}_{mn} \nabla_n \overline{b}_j + V^{ij}_m \overline{b}_j$$

Note that

- Eddy diffusivities and wave drifts mix different components (flux of P depends on gradient of Z).
- If R has a negative lobe, the biological diffusivities can be larger than that of a passive scalar
- The quasi-equilibrium approximation

$$\mathcal{B}_{ij} b'_j = \mathbf{u}' \cdot \nabla \overline{b}_i$$

works reasonably well in the upper water column. In particular

$$\mathcal{B}_{21} = g\overline{Z} \exp(-\nu \overline{P}) > 0 \quad . \quad \mathcal{B}_{22} = 0$$

so that

$$P' = \frac{1}{\mathcal{B}_{21}} \mathbf{u}' \cdot \nabla \overline{Z} \quad \text{unlike} \quad C' = -\boldsymbol{\xi} \cdot \nabla \overline{C}$$

`Demos, Page 5:` complex diffusion <*transport coeff: display -geometry +0+0 -bordercolor white -border 20x20 -rotate 90 ~glenn/12.822t/graphics/t0.ps*> up Z grad flux of Pt1.ps <*quasiequilibrium fluxes: display -geometry +0+0 -bordercolor white -border 20x20 -rotate 90 ~glenn/12.822t/graphics/t1a.ps*> down-gradient Kpp,KZZt2.ps

Eulerian-Lagrangian

If $\kappa = 0$, we can relate the relevant form of the Eulerian covariance

$$R_{mn}(\mathbf{x}, t|\mathbf{x}', t') = \overline{u'_m(\mathbf{x}, t)G(\mathbf{x}, t|\mathbf{x}', t')u_n(\mathbf{x}', t')}$$

to Taylor's form. The Greens' function equation

$$\frac{\partial}{\partial t}G + \mathbf{u}(\mathbf{x}, t) \cdot \nabla G = \delta(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

has a solution

$$G(\mathbf{x}, t|\mathbf{x}', t') = \delta\left(\mathbf{x} - \mathbf{X}(t|\mathbf{x}', t')\right)$$

where

$$\frac{\partial}{\partial t}\mathbf{X}(t|\mathbf{x}', t') = \mathbf{u}(\mathbf{X}, t) \quad , \qquad \mathbf{X}(t'|\mathbf{x}', t') = \mathbf{x}'$$

gives the Lagrangian position of the particle initially at $\mathbf{x}'$ at time t'. But it is more convenient to back up along the trajectory and let

$$G(\mathbf{x}, t|\mathbf{x}', t') = \delta(\mathbf{x}' - \boldsymbol{\xi}(t - t'|\mathbf{x}, t))$$

where the particle at $\boldsymbol{\xi}$ at time t' passes $\mathbf{x}$ at time t (and takes a time τ for this tranistion). Thus the $\boldsymbol{\xi}$'s give the starting position, which, for stochastic flows varies from realization to realization. We can solve

$$\frac{\partial}{\partial \tau}\boldsymbol{\xi}(\tau|\mathbf{x}, t) = -\mathbf{u}(\boldsymbol{\xi}(\tau|\mathbf{x}, t), t - \tau) \quad , \qquad \boldsymbol{\xi}(0|\mathbf{x}, t) = \mathbf{x}$$

for $\tau = 0$ to $\tau = t - t'$ to find $\boldsymbol{\xi}$.

We can now define the generalization of the Lagrangian correlation function used by Taylor

$$\begin{aligned} R_{mn}(t - t', \mathbf{x}) &= \int d\mathbf{x}'\overline{u'(\mathbf{x}, t)G(\mathbf{x}, t|\mathbf{x}', t')u'(\mathbf{x}', t')} \\ &= \overline{u'_m(\mathbf{x}, t)u'_n(\boldsymbol{\xi}(t - t'|\mathbf{x}, t), t - (t - t'))} \end{aligned}$$

or

$$R_{mn}(\tau, \mathbf{x}) = \overline{u'_m(\mathbf{x}, t)u'_n(\boldsymbol{\xi}(\tau|\mathbf{x}, t), t - \tau)}$$

For homogeneous, stationary turbulence (on the scales intermediate between the eddies and the mean), this will be equivalent to Taylor's

$$R_{mn}(\tau) = \overline{u'_m(\mathbf{X}(t' + \tau|\mathbf{x}', t'), t' + \tau)u'_n(\mathbf{x}', t')}$$

but we include inhomogeneity and (for, general G, diffusion).