Particle Dispersion

Random Flight – Lagrangian dispersion

As an example, we examine the random flight model, which assumes that the accelerations have a stochastic component and use Newton's equations

$$\begin{aligned} d\mathbf{X} &= \mathbf{V}dt \\ d\mathbf{V} &= \mathbf{A}dt + \beta d\mathbf{R} \end{aligned}$$

where $\mathbf{A}$ is the acceleration produced by deterministic (or large-scale) forces. We include random accelerations with the random increment $d\mathbf{R}$ satisfying $\langle dR_i \, dR_j \rangle = \delta_{ij} dt$.

As examples, consider a drag law for the acceleration

$$\mathbf{A} = -r(\mathbf{V} - \mathbf{u})$$

with $\mathbf{u}$ being the water velocity. The dispersion is determined by β and r; from the equations, we can show that

$$\begin{aligned} \langle V_i \rangle &\to u_i \\ \langle (V_i - u_i)(V_j - u_j) \rangle &\to \frac{\beta^2}{2r}\delta_{ij} \\ \langle X_i(t) X_j(t) \rangle &\to \langle X_i(0) X_j(0) \rangle + \frac{\beta^2}{r^2}\delta_{ij} t \end{aligned}$$

The latter corresponds to a diffusivity of $\kappa = \beta^2/2r^2$.

- Area grows like $4\kappa t$ ($6\kappa t$ in 3-D)
- Velocity variance is $r\kappa$

`Demos, Page 1: Random flight` *<dispersion>* *<mean sq displacement>*

Taylor dispersion

In 1922, Taylor described the dispersion under the assumption that the Lagrangian velocity had a known covariance structure. He considered just

$$\frac{\partial}{\partial t}\mathbf{X} = \mathbf{V}(t)$$

We find that

$$\frac{\partial}{\partial t} X_i X_j = V_i X_j + X_i V_j$$

and, in the ensemble average,

$$\frac{\partial}{\partial t}\langle X_i X_j\rangle = \langle V_i X_j\rangle + \langle X_i V_j\rangle$$

If we substitute

$$\mathbf{X} = \mathbf{X}_0 + \int_0^t \mathbf{V}(t')dt'$$

and look at the case where $\langle \mathbf{V}\rangle = 0$ and the flow is stationary, we have

$$\frac{\partial}{\partial t}\langle X_i X_j\rangle = \int_0^t dt'\ R_{ij}^L(t') + R_{ji}^L(t')$$

where R_{ij}^L is the covariance of the Lagrangian velocities

$$R_{ij}^L(t) = \langle V_i(t_0 + t) V_j(t_0)\rangle$$

For isotropic motions $R_{ij}^L(t) = U^2 R^L(t)\delta_{ij}$ with $R^L(t)$ being the autocorrelation function; the change in x-variance is given by

$$\frac{\partial}{\partial t}\langle X^2\rangle = 2U^2 \int_0^t R^L(t)$$

From this formula, we see that

- For short times,
$$\langle X^2\rangle = U^2 t^2$$
- For long times, if the integral $T_{int} = \int_0^\infty R^L(t)dt$ is finite and non-zero,
$$\langle X^2\rangle = 2U^2 T_{int} t$$

Relation to diffusivity

Consider the diffusion of a passive scalar

$$\frac{\partial}{\partial t}C = \kappa\nabla^2 C$$

and define moments of the distribution

$$\langle x^n \rangle = \frac{\iint x^n C}{\iint C}$$

Integrating the diffusion equation gives conservation of the total scalar, under the assumtion that the initial distribution is compact and the values decay rapidly at infinity

$$\frac{\partial}{\partial t}\iint C = \kappa \oint \hat{\mathbf{n}} \cdot \nabla C = 0$$

The first moment gives

$$\frac{\partial}{\partial t}\iint xC = \kappa \iint x\nabla^2 C = \kappa \iint \nabla \cdot x\nabla C - \frac{\partial}{\partial x}C = \kappa \oint \hat{\mathbf{n}} \cdot [x\nabla C - \hat{\mathbf{x}}C] = 0$$

so that $\frac{\partial}{\partial t}\langle x \rangle = 0$. (This result would be different if there were flow as well.)

The second moment

$$\frac{\partial}{\partial t}\iint x^2 C = \kappa \iint x^2\nabla^2 C = \kappa \iint \nabla \cdot [x^2\nabla C - 2x\hat{\mathbf{x}}C] + 2C = 2\kappa \iint C$$

implies that

$$\frac{\partial}{\partial t}\langle x^2 \rangle = 2\kappa$$

Thus we can identify the effective diffusivity

$$\kappa = U^2 T_{int}$$

Small amplitude motions

If we assume that the scale of a typical particle excursion over time T_{int} is small compared to the scale over which the flow varies, we can relate the Lagrangian and Eulerian statistics. The displacement $\xi_i = X_i(t) - X_i(0)$ satisfies

$$\frac{\partial}{\partial t}\xi_i = u_i(\mathbf{x}+\xi, t) \simeq u_i(\mathbf{x}, t) + \xi_j \frac{\partial}{\partial x_j} u_i(\mathbf{x}, t) + \ldots$$

and we can substitute the lowest order solution

$$\xi_i(t) = \int_0^t dt' u_i(\mathbf{x}, t')$$

into the second term above to write

$$\frac{\partial}{\partial t}\xi_i = u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\int_0^t u_j(\mathbf{x}, t') u_i(\mathbf{x}, t)$$

and average, recognizing that the mean Lagrangian velocity is just $\langle \frac{\partial}{\partial t}\xi_i \rangle$:

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t \langle u_i(\mathbf{x}, t) u_j(\mathbf{x}, t') \rangle$$

For simplicity, we assume that the turbulent velocities are large compared to the mean; then this becomes

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t R_{ij}(\mathbf{x}, t-t') = \langle u_i \rangle + \frac{\partial}{\partial x_j}\int_0^t d\tau R_{ij}(\mathbf{x}, \tau)$$

Let us assume that the integrals with respect to τ exist and split the covariance into its symmetric and antisymmetric parts

$$\langle u_i^L \rangle = \langle u_i \rangle + \frac{\partial}{\partial x_j} D_{ij}^s(\mathbf{x}) + \frac{\partial}{\partial x_j} D_{ij}^a$$

with

$$K_{ij} \equiv D_{ij}^s = \frac{1}{2}\int_0^\infty R_{ij}(x, \tau) + R_{ji}(\mathbf{x}, \tau) \quad , \quad D_{ij}^a = \frac{1}{2}\int_0^\infty R_{ij}(x, \tau) - R_{ji}(\mathbf{x}, \tau)$$

We can write an arbitrary antisymmetric tensor in terms of the unit antisymmetric tensor

$$D_{ij}^a = -\epsilon_{ijk}\Psi_k$$

so that the contribution to the Lagrangian velocity is

$$u_i^S = -\epsilon_{ijk}\frac{\partial}{\partial x_j}\Psi_k \quad , \quad \mathbf{u}^S = -\nabla \times \Psi$$

Note that the antisymmetric part of the contribution to the Lagrangian velocity is nondivergent:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}D^a_{ij}(\mathbf{x}) = \nabla \cdot \mathbf{u}^S = 0$$

Thus the Lagrangian mean velocity has contributions from the mean Eulerian flow, from the Stokes' drift, and a term which tends to move into regions of higher diffusivity

$$\langle u_i^L \rangle = \langle u_i \rangle + u_i^S + \frac{\partial}{\partial x_j}K_{ij}(\mathbf{x})$$

We will discuss the meanings of these terms in more detail next.

Chaotic advection and Stokes' drift

We start with the basic wave

$$\psi = \frac{\epsilon}{\pi}\sin(\pi[x-t])\sin(\pi y)$$

and add a small amount of a second wave

$$\psi = \sqrt{1-16\alpha^2}\frac{\epsilon}{\pi}\sin(\pi[x-t])\sin(\pi y) + \alpha\frac{\epsilon}{\pi}\sin(4\pi[x-c_1 t])\sin(4\pi y)$$

Demos, Page 5: psi <*alpha=0*> <*alpha=0.01*> <*alpha=0.1*>

We look at the particle trajectories by solving the Lagrangian equations as above

$$\frac{\partial}{\partial t}\boldsymbol{\xi} = \mathbf{u}(\mathbf{x}+\boldsymbol{\xi}, t)$$

Let's begin with the simplest case without the second wave. For small ϵ (which is the ratio of the flow speed to the phase speed, we can find an approximate solution (as before) by iterating

$$\begin{aligned}\frac{\partial}{\partial t}\xi_i &\simeq u_i(\mathbf{x}, t') + \xi_i\frac{\partial}{\partial x_j}u_i(\mathbf{x}, t) + \ldots \\ &\simeq u_i(\mathbf{x}, t) + \frac{\partial}{\partial x_j}\int_0^t u_j(\mathbf{x}, t')u_i(\mathbf{x}, t)dt'\end{aligned}$$

The mean Lagrangian drift is therefore

$$\overline{\frac{\partial}{\partial t}\xi_i} = \frac{\partial}{\partial x_j}\int_0^t R_{ij}(\mathbf{x}, \tau)d\tau$$

For the primary wave

$$R_{ij}(\tau) = \frac{\epsilon^2}{2\pi}\begin{pmatrix}\cos\pi\tau\cos^2\pi y & -\cos\pi\tau\sin\pi y\cos\pi y \\ \sin\pi\tau\sin\pi y\cos\pi y & \cos\pi\tau\sin^2\pi y\end{pmatrix}$$

and the drift is

$$u_L = \overline{\frac{\partial}{\partial t}\xi_1} = \frac{\epsilon^2}{2}\cos(2\pi y)[1-\cos(\pi t)]$$

$$v_L = \overline{\frac{\partial}{\partial t}\xi_2} = \frac{\epsilon^2}{2}\sin(2\pi y)\sin\pi t$$

Note that there is a mean drift

$$\overline{u_L} = \frac{\epsilon^2}{2}\cos(2\pi y)$$

prograde on the walls and retrograde in the center. `Demos, Page 5: drift` *<amp=0.2>* *<amp=0.2 comoving>* *<amp=1.0>* *<amp=1.0 comoving>* *<stokes drift>* *<mean>*

FINITE AMPLITUDE

In the frame of reference of the wave ($\mathbf{X}' = \mathbf{X} - \mathbf{c}t$)

$$\frac{\partial}{\partial t}\mathbf{X}' = \mathbf{u}(\mathbf{X}') - \mathbf{c} = \hat{\mathbf{z}} \times \nabla(\psi + cy)$$

Thus particles simply move along the streamlines. At some Lagrangian period T_L, the particle will have moved one period to the left so that

$$X'(T_L) = X(0) - \lambda = X(T_L) - cT_L \quad \Rightarrow \quad u_L = \frac{X(T_L) - X(0)}{T_L} = c - \frac{\lambda}{T_L} = c\,(1 - \frac{T_E}{T_L})$$

Stokes drifts occur when the Lagrangian period differs from the Eulerian period. Trapped particles have

$$X'(T_L) = X(0) = X(T_L) - cT_L \quad \Rightarrow \quad u_L = \frac{X(T_L) - X(0)}{T_L} = c$$

Back to chaotic advection...

When we have α non-zero, the trajectories become less regular in the vicinity of the stagnation points. A line of particles approaching the point begins to fold, with some fluid crossing into the interior and some being ejected. Which way a parcel goes depends on the phase of the perturbing wave as it nears the stagnation point.

`Demos, Page 6: lobe dynamics` *<alpha 0.008>*

We can look at Poincaré sections (snapshots at the period of the perturbing wave) at various amplitudes to see the mixing regions `Demos, Page 6: poincare sections` *<alpha=0>* *<alpha=0.002>* *<alpha=0.004>* *<alpha=0.008>* *<alpha=0.016>* *<alpha=0.032>* *<alpha=0.064>* *<alpha=0.128>*

The mixing across the channel is still blocked for α small enough < 0.05 so the mixing is still diffusion-limited, although some gain is realized by enhanced flux out of the wall and a decrease in the width of the blocked region.

`Demos, Page 6: Continuum` *<steady>* *<weak>* *<strong>*

References

Flierl, G.R. (1981) Particle motions in large amplitude wave fields. *Geophys. Astrophys. Fluid Dyn.*, **18**, 39-74.

Pierrehumbert, R.T. (1991) Chaotic mixing of tracer and vorticity by modulated travelling Rossby waves. *Geophys. Astrophys. Fluid Dyn.*, **58**, 285-319.

Tracer fluxes

Next time, we'll see that the mean concentration (in appropriate limits) satisfies

$$\frac{\partial}{\partial t}\langle C\rangle = -\nabla\cdot\left[\langle \mathbf{u}C'\rangle - \kappa\nabla C\right]$$

and

$$\langle u_i(t)C'(t)\rangle = -\left[\int_0^t dt'\langle u_i(t)u_j(t')\rangle\right]\frac{\partial}{\partial x_j}\langle C\rangle = \left[u_i^S - K_{ij}\frac{\partial}{\partial x_j}\right]\langle C\rangle$$

With this form, we can see that the Stokes' drift does not alter tracer variance (or maxima), while the K_{ij} term tends to reduce the maxima and the tracer variance. Thus it is appropriate to think of K_{ij} as a diffusivity tensor.

In addition, we note that for variable K, the center of mass of the tracer satsfies

$$\begin{aligned}\frac{\partial}{\partial t}X_i\int C &= -\int x_i\nabla_j\left[(\langle u_j\rangle + u_j^S - K_{jk}\nabla_k)C\right]\\ &= \int \langle u_i\rangle C + u_i^S C - K_{ik}\nabla_k C\\ &= \int \langle u_i\rangle C + u_i^S C + [\nabla_k K_{ik}]C\\ &\simeq \left[\langle u_i\rangle + u_i^S + \nabla_k K_{ik}\right]\int C\\ &\simeq \langle u_i^L\rangle\int C\end{aligned}$$

so that the center of mass of the tracer (for narrow distributions) indeed moves with the mean Eulerian flow, the Stokes' drift and the up-diffusivity-gradient term.

Random Rossby Waves

Consider a randomly-forced Rossby wave in a channel:

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi + y) = \gamma \mathrm{Re}[r(t)e^{\imath kx}]\sin(\ell y) - \gamma\nabla^2\psi$$

where r is randomly distributed on a disk of radius r_0. This gives a streamfunction

$$\psi = \mathrm{Re}[a(t)e^{\imath kx}]\sin(\ell y)$$

with

$$\frac{d}{dt}a + (\gamma + \imath\,\omega)a = \frac{\omega\gamma}{\beta k}r$$

and $\omega = -\beta k/(k^2+\ell^2)$.

$$a = \frac{\gamma}{2}\int_{-\infty}^{t} d\tau e^{-(\gamma-\frac{1}{2}\imath)\tau}r(t-\tau)$$

`Demos, Page 8:  random field` *<rand field>* *<mean>*

From this, we find

$$\overline{\psi(x,y,t)\psi(x',y',t')} =$$

$$\frac{U_0^2}{2\ell^2}e^{-\gamma(t-t')}\cos[k(x-x') - \omega(t-t')]\sin(\ell y)\sin(\ell y')$$

$$R_{mn}(\tau) = \frac{1}{2}U_0^2 e^{-\gamma\tau}\begin{pmatrix} \cos\omega\tau\cos^2\ell y & \frac{k}{\ell}\sin\omega\tau\sin\ell y\cos\ell y \\ -\frac{k}{\ell}\sin\omega\tau\sin\ell y\cos\ell y & \frac{k^2}{\ell^2}\cos\omega\tau\sin^2\ell y \end{pmatrix}$$

Hence the mean drift is given by

$$\mathbf{u}^L = \frac{r_0^2\gamma}{64(\gamma^2+\frac{1}{4})}\big(-\cos(2y)\ ,\ \ 2\gamma\sin(2y)\big)$$

or

$$\mathbf{u}^L = \frac{KE}{\gamma^2+\frac{1}{4}}\big(-\cos(2y)\ ,\ \ 2\gamma\sin(2y)\big)$$

with $KE = \frac{1}{2}\overline{u^2+v^2}$.

The Stokes drift term is

$$\mathbf{u}^S = \frac{KE}{\gamma^2+\frac{1}{4}}\big(-\cos(2y)\ ,\ \ 0\big)$$

while the diffusivity tensor is

$$K_{ij} = 2\gamma\frac{KE}{\gamma^2+\frac{1}{4}}\begin{pmatrix}\cos^2(y) & 0 \\ 0 & \sin^2(y)\end{pmatrix}$$

`Demos, Page 8:  stokes drift` *<lin vs act sd>* *<mean drift>*

Conclusions:

- Rossby waves cause mean westward drifts at the edges and eastward drifts in the center.
- Eddy diffusivities are spatially variable and anisotropic.