QG Turbulence and Waves

The quasigeostrophic equation contains a number of essential features of large scale geophysical flows, while retaining some of the simplicity of 2D flow. We assume that the system is rapidly rotating and hydrostatic, so that the vertical vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + \mathbf{u}\cdot\nabla(f+\zeta) - (f\hat{\mathbf{z}}+\boldsymbol{\zeta})\cdot\nabla w = diss$$

with $\boldsymbol{\zeta} = (-v_z, u_z, \zeta)$ and the β-plane approximation $f = f_0 + \beta y$. We assume that ζ is small compared to f ($Ro = \zeta/f \ll 1$) though it may be similar in size to $\delta f/f = \beta L/f$. In that case the term $\boldsymbol{\zeta}\cdot\nabla w$ is order ζ/f compared to $f\frac{\partial}{\partial z}w$. Thus the vortex stretching is dominantly associated with fluid columns suffering extension along the rotation axis.

$$\frac{\partial}{\partial t}\zeta + \mathbf{u}\cdot\nabla(f+\zeta) - f\frac{\partial}{\partial z}w = diss$$

We can in general represent a nondivergent flow as

$$\mathbf{u} = -\nabla\times(\psi_1, \psi_2, \psi)$$

and we choose a gauge such that $\frac{\partial}{\partial x}\psi_1 + \frac{\partial}{\partial y}\psi_2 = 0$ so that $\psi_1 = \frac{\partial}{\partial y}\phi$ and $\psi_2 = -\frac{\partial}{\partial x}\phi$. Then

$$\mathbf{u} = \hat{\mathbf{z}}\times\nabla\psi - \nabla\frac{\partial\phi}{\partial z} \quad , \quad w = \nabla^2\phi \quad , \quad \zeta = \nabla^2\psi$$

For near-geostrophic balance,

$$\mathbf{u} = \frac{1}{f}\hat{\mathbf{z}}\times\nabla p$$

the divergence $\frac{\partial\phi}{\partial z}$ is order $\beta L/f$ times the vorticity. Thus the vorticity equation becomes

$$\frac{\partial}{\partial t}\zeta + J(\psi, \zeta + \beta y) = f\frac{\partial}{\partial z}w + diss$$

with $J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$ and $\zeta = \nabla^2\psi$.

The buoyancy equation can also be simplified by neglecting the divergent part of the horizontal flow and noting that $\frac{\partial}{\partial z}b/N^2$ is order $(\zeta/f)(f^2L^2/N^2H^2$; for synoptic/ mesoscale flows the last factor is order one. Thus

$$\frac{\partial}{\partial t}b + J(\psi, b) + wN^2 = heat$$

Using $b = \frac{\partial}{\partial z}p = f\frac{\partial}{\partial z}\psi$ and combining gives the QG equation

$$\frac{\partial}{\partial t}q + J(\psi, q) = diss/heat$$

with

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial}{\partial z}\psi + \beta y$$

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The QG equations determine the evolution of a scalar property, the approximate potential vorticity q, under advection by the horizontal flow $\mathbf{u} = \hat{\mathbf{z}} \times \nabla\psi$. Although the movement of PV is treated two-dimensionally at a given depth, the flow is related to the PV structure at nearby depths.

Conserved properties

The QG equations preserve energy

$$E = \frac{1}{2} \int d^3\mathbf{x} \, |\nabla\psi|^2 + \frac{1}{N^2} \left| \frac{\partial\psi}{\partial\mathbf{z}} \right|^2 = \int d^3\mathbf{x} \, \mathcal{E}$$

and potential enstrophy

$$Z_P = \frac{1}{2} \int d^3\mathbf{x} \, q^2$$

Indeed, they conserve the average of any function of the PV, not just the square, so that we have to worry about whether or not the energy and entrophy tell the whole story.

The β term also can have important consequences, depending on the boundaries. If we represent

$$q = \mathcal{L}\psi + \beta y$$

then

$$Z_P = \frac{1}{2} \int d^3\mathbf{x} \, |\mathcal{L}\psi|^2 + \beta \int d^3\mathbf{x} \, y\mathcal{L}\psi + const.$$

In the doubly-periodic case with uniform buoyancy on the top and bottom boundaries ($\frac{\partial}{\partial z}\psi = 0$) the only surviving term is

$$Z = \frac{1}{2} \int d^3\mathbf{x} \, |\mathcal{L}\psi|^2$$

the one which can be thought of as $\int d^3\mathbf{x} \, K^2\mathcal{E}$. We'll talk about other cases later.

Charney's spectrum

Charney (1971) argues that for small enough scales in the interior of the atmosphere, we can treat N^2 as constant, rescale $z^* = Nz/f$, and transform the $\mathcal{L}$ operator into $\nabla_3^2\psi$. All of the arguments for upscale energy transfer and downscale enstrophy transfer apply, so that the spectrum should be

$$E(k) \sim K^{-3} = (k^2 + \ell^2 + m^{*2})^{-3/2} = (k^2 + \ell^2 + m^2 f^2/N^2)^{-3/2}$$

just as in the 2-D case. In addition, the theory predicts equipartition of energy among the u, v, and bf/N fields. Demos, Page 2: Data <Gage and Nastrom, 1986>

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Fjörtöft's argument

Fjörtöft's (1953) argument can also be applied to the 3D QG flow problem. Suppose we have unit energy at a **net** wavenumber K such that

$$\mathcal{L}\psi = -K^2\psi$$

and we wish to transfer it elsewhere through inviscid interactions. Let a fraction α_1 go to larger scales $(K/2)$ and α_2 to smaller scales $2K$. Then our energy and enstrophy pictures look like

Wavenumber	$K/2$	K	$2K$
Init. energy	0	1	0
Init. enstrophy	0	K^2	0
Final energy	α_1	$1 - \alpha_1 - \alpha_2$	α_2
Final enstrophy	$K^2\alpha_1/4$	$K^2(1 - \alpha_1 - \alpha_2)$	$K^2 4\alpha_2$

If we conserve both energy and enstrophy by this interaction (i.e., we're in an inertial range), we find $\alpha_1 = 4\alpha_2$ so that

Wavenumber	$K/2$	K	$2K$
Init. energy	0	1	0
Init. enstrophy	0	K^2	0
Final energy	$4\alpha_2$	$1 - 5\alpha_2$	α_2
Final enstrophy	$K^2\alpha_2$	$K^2(1 - 5\alpha_2)$	$K^2 4\alpha_2$

More energy is transferred to large scales and more enstrophy to small scales. Indeed the center of the energy is now at wavenumber $K(1 - \alpha_2)$ and the center of the enstrophy is at $K(1 + 3.5\alpha_2)$. Remember that in the cascade to larger scale, the vertical scale can increase – the flow can become more barotropic.

Beta effects

Note that these arguments make no mention of the variation of the Coriolis parameter with latitude, β. While it is true that the β-effect does not make the QG equations inhomogeneous (the full equations or the shallow water equations are a different matter), it does make the dynamics anisotropic. Rotation by 90 degrees alters the form of q. Rhines showed that turbulence on the β-plane has a profoundly different charater, developing zonal bands of flow. He used the barotropic vorticity equation (the $\frac{\partial}{\partial z} = 0$ case of the QG equation, though the BTVE is actually an exact representation of 2-D motion on a β-plane)

$$\frac{\partial}{\partial t} q + J(\psi, q) = diss \quad , \quad q = \nabla^2 \psi + \beta y$$

The dynamics now includes both turbulence and waves riding on the large-scale potential vorticity gradient β. The evolution of the flow becomes at some point a problem of interacting waves rather than multiple-scale energy transfers.

We can see that this will happen at some scale by considering the parameter measuring nonlinear versus wave effects – the wave steepness $S = U/c$. Since the phase speed for Rossby waves is $-\beta/k^2$, $S = Uk^2/\beta$. For a k^{-3} energy spectrum, we have the velocities proportional to k^{-1} and the steepness behaves like k. Therefore, we expect the β-effect will have little influence on the short waves, but that the long waves will have restoring forxes which are as significant as the turbulent transfers. The scale at which this transition occurs should be when the steepness is order one, or $L \sim \sqrt{U/\beta} = E^{1/4}\beta^{-1/2}$.

Alternatively, we could view the effects of the turbulence as mixing the PV and attempting to homogenize it. But this can only be done over narrow latitude bands. Suppose we start with an initial eddy energy density E. If we homogenize the PV over width W, the zonal mean flow looks like

$$-\frac{\partial}{\partial y} U + \beta y = const. = 0 \quad \text{for} \quad -W/2 < y < W/2$$

so that $U = \beta y^2/2 - \beta W^2/24$. The energy density for this flow is

$$\frac{1}{W} \int_{-W/2}^{W/2} dy\, U^2 = \frac{\beta^2 W^4}{1440}$$

If we used all the initial energy and put it into zonal flow, we'd have

$$W = 38 E^{1/4}\beta^{-1/2}$$

Two things prevent this from happening; not all of the energy goes into the waves and the interactions become very slow as wave processes dominate.

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Baroclinicity

The transfer to large scale occurs in both horizontal and vertical directions. Therefore, we expect the energy in the gravest vertical mode ($F = 1$, $\lambda_0 = 0$) to dominate after a while. We can expand $q' = \mathcal{L}\psi$ and ψ in the vertical eigenfunctions

$$q' = q_m(x, y, t)F_m(z) \quad , \quad \psi = \psi_m(x, y, t)F_m(z)$$

with

$$q_m = \nabla_2^2 \psi_m - \lambda^2 \psi_m$$

The F_m functions are the eigenfunctions of the vertical operator

$$\frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial}{\partial z}F_m = -\lambda_m^2 F_m \quad , \quad \frac{1}{H}\int_0^H F_n(z)F_m(z) = \delta_{nm}$$

Then the energy is

$$E = -\iint q_m \psi_m = E_0 + E_1 + E_2...$$

Demos, Page 5: Two vertical mode case `<pv>` `<psi>` `<energies>`

Spectral space transfers

Let us transform the streamfunction to wavenumber space

$$\psi = \hat{\psi}(k, \ell, m, t)\exp(\imath kx + \imath \ell y)F_m(z)$$
$$q' = \hat{q}(k, \ell, m, t)\exp(\imath kx + \imath \ell y)F_m(z)$$

We introduce a shorthand $\psi_j = \hat{\psi}(k, \ell, \lambda_m, t)$ so that each different subscript j corresponds to a different set of $\{k, \ell, m\}$ values. The streamfunction is related to the potential vorticity by

$$q_j = -(k_j^2 + \ell_j^2 + \lambda_{m_j}^2)\psi_j \equiv -K_j^2 \psi_j$$

Now we can project out the equation for the amplitude of one mode by multiplying the equation by $F_{m_2}(z)\exp(-\imath \mathbf{k}_2 \cdot \mathbf{x})$ and volume averaging

$$K_2^2 \frac{\partial}{\partial t}\psi_2 = \imath \beta k_2 \psi_2 + \sum \sum_{\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0} \xi_{m_1 m_2 m_3}(k_1 \ell_3 - k_3 \ell_1)K_3^2 \psi_1^* \psi_3^*$$

or

$$K_2^2 \frac{\partial}{\partial t}\psi_2 = \imath \beta k_2 \psi_2 + \sum \xi_{m_1 m_2 m_3}(k_1 \ell_2 - k_2 \ell_1)(K_1^2 - K_3^2)\psi_1^* \psi_3^*$$

with the definition $\mathbf{k}_3 = -\mathbf{k}_1 - \mathbf{k}_2$.

Let us look at one set of wavenumbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and choose the labelling such that $K_1^2 < K_2^2 < K_3^2$. The dynamics of this triad is given by

$$K_1^2 \frac{\partial}{\partial t}\psi_1 = \imath\beta k_1\psi_1 + \xi_{m_1 m_2 m_3}(k_1\ell_2 - k_2\ell_1)(K_3^2 - K_2^2)\psi_3^*\psi_2^*$$

$$K_2^2 \frac{\partial}{\partial t}\psi_2 = \imath\beta k_2\psi_2 + \xi_{m_1 m_2 m_3}(k_1\ell_2 - k_2\ell_1)(K_1^2 - K_3^2)\psi_1^*\psi_3^*$$

$$K_3^2 \frac{\partial}{\partial t}\psi_3 = \imath\beta k_3\psi_3 + \xi_{m_1 m_2 m_3}(k_1\ell_2 - k_2\ell_1)(K_2^2 - K_1^2)\psi_2^*\psi_1^*$$

This triad conserves energy and enstrophy internally

$$\frac{\partial}{\partial t}\sum K_j^2|\psi_j|^2 = \sum E_j = 0$$

$$\frac{\partial}{\partial t}\sum K_j^4|\psi_j|^2 = \sum K_j^2 E_j = 0$$

From the triad equations, we also have

$$\frac{\partial}{\partial t}E_1 = -\frac{K_3^2 - K_2^2}{K_3^2 - K_1^2}\frac{\partial}{\partial t}E_2$$

$$\frac{\partial}{\partial t}E_3 = -\frac{K_2^2 - K_1^2}{K_3^2 - K_1^2}\frac{\partial}{\partial t}E_2$$

Energy leaving component 2 will transfer into both 1 and 3; when it does so, the average scale $\left(\sum K_j E_j / \sum E_j\right)^{-1}$ increases; however, only the triads with $K_3^2 - K_2^2 > K_2^2 - K_1^2$ will actually put more energy into the larger scale mode than the smaller scale one.

    Demos, Page 6: Example <(0,0,0) triads> <sigma>

    STABILITY: If we start with energy in the second component, we can calculate the rate at which it goes to other components by looking at the growth rate. We assume

$$\psi_2 = A_2\exp(\imath\beta k_2/K_2^2)$$

and

$$\psi_1 = A_1(t)\exp(-\imath\beta k_2/2K_2^2)$$
$$\psi_3 = A_3(t)\exp(-\imath\beta k_2/2K_2^2)$$

so that the perturbation problem becomes

$$\frac{\partial}{\partial t}A_1 = -\imath(\omega_1 + \frac{\omega_2}{2})A_1 + \xi(k_1\ell_2 - k_2\ell_1)\frac{K_3^2 - K_2^2}{K_1^2}A_3^*A_2^*$$

$$\frac{\partial}{\partial t}A_3^* = \imath(\omega_3 + \frac{\omega_2}{2})A_3^* + \xi(k_1\ell_2 - k_2\ell_1)\frac{K_2^2 - K_1^2}{K_3^2}A_2 A_1$$

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The growth rates are determined by

$$[\sigma + \imath(\omega_1 + \frac{\omega_2}{2})][\sigma - \imath(\omega_3 + \frac{\omega_2}{2})] = \xi^2(k_1\ell_2 - k_2\ell_1)^2 \frac{K_3^2 - K_2^2}{K_1^2} \frac{K_2^2 - K_1^2}{K_3^2} |A_2|^2$$

when the amplitude is small, the growth rate will be nonzero only in the regions where

$$\omega_1 + \frac{\omega_2}{2} = -\omega_3 - \frac{\omega_2}{2} \quad or \quad \omega_1 + \omega_2 + \omega_3 = 0$$

 Demos, Page 6: Resonance <phi=0;beta=0> <beta=0.5> <beta=1> <beta=2>
<beta=5>
    Demos, Page 7: Resonance-angle   <phi=30;beta=0>   <phi=45;beta=0>
<phi=60;beta=0> <phi=30;beta=2>   <phi=45;beta=2>   <phi=60;beta=2>
    Demos, Page 7: Resonance-modes   <(0,0,0)>   <(1,1,0)>   <(0,1,1)>
<(1,1,0)>   <(0,1,1)>