Convection problem set

1. Hele-Shaw Cell Convection

The Hele-Shaw cell has a thin layer of fluid confined between two glass or plexiglass plates at $y = 0$ and $y = \delta$. It allows us to study real analogues to 2-D flows.

1) Assume that u and w have the characteristic parabolic profile in y

$$u(x, y, z, t) = u(x, z, t)f(y) \quad , \qquad w(x, y, z, t) = w(x, z, t)f(y) \quad , \qquad f(y) = 6\frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)$$

and that pressure and buoyancy are independent of y and $v = 0$. Take the y average of the scaled equations

$$\begin{aligned}
\frac{1}{Pr}\frac{D}{Dt}\mathbf{u} &= -\nabla p + Ra\, b\,\hat{\mathbf{z}} + \nabla^2\mathbf{u} \\
\nabla \cdot \mathbf{u} &= 0 \\
\frac{D}{Dt}b &= w + \nabla^2 b
\end{aligned}$$

(b being the deviation from $\overline{b} = -z$). Show that the averaged momentum equation reduces to

$$\frac{12}{\delta^2}\mathbf{u} = -\nabla p + Ra\, b\,\hat{\mathbf{z}}$$

when δ is very small (thin fluid layer) and the Rayleigh number is big.

2) Define the streamfunction and the y-component of the vorticity. Eliminate the pressure and write coupled equations for ψ and b.

3) Solve the linear stability problem and find the critical Rayleigh number assuming free-slip conditions on top and bottom,

4) Consider now the nonlinear problem. We could solve for weakly supercritical conditions by assuming $\epsilon = Ra - Ra_c$ is small and expanding in suitable powers. However, it is easier to follow Lorenz and examine a truncated systems. Derive the equivalents to the Lorenz equations for this system. You can use the same expansions for b and the vorticity as in the notes. Show that the resulting equations have a simple bifurcation from the motionless state to a stable steady state.

2. Line Plume

A line plume is driven by a steady buoyancy flux per unit length G_0 (units of $m^3 s^{-3}$) on a line which extends infinitely far in one horizontal direction y. The resultant plume is therefore 2-dimensional, with no variation in the y-direction. Entrainment into the plume occurs only in the x-direction. In this problem, we'll examine the derivation of the plume equations and the solutions.

1) Let the turbulent plume extend from $-r(z) < x < r(z)$. If the boundary were an impermeable surface, then we would have

$$u = w\frac{\partial r}{\partial z}$$

but entrainment adds an extra flow which we'll call $-u_e$ (negative since the entrainment is going inwards

$$u = w\frac{\partial r}{\partial z} - u_e$$

Integrate the two-D continuity equation from 0 to r^+ and show that the averaged vertical velocity

$$W = \frac{1}{r}\int_0^r dxw$$

satisfies

$$\frac{\partial}{\partial z}Wr = u_e$$

2) Now consider the flux of buoyancy, integrating from 0 to r^+ where the outside buoyancy value is b_e. Assume you can replace the average of wb with W times B, the average of b.

3) Finally treat the vertical momentum. The additional assumption is that the pressure is the same as the environment.

4) Now look for similarity solutions: assume $W \sim z^n$, $r \sim z^m$ and find the solutions given a uniform background buoyancy $b_e = 0$. Make the entrainment approximation $u_e = \alpha W$, where α is a constant.

5) Suppose you can assess the effects of rotation by finding the height z_f beyond which the plume Rossby number $Ro = W/(fr)$ is less than 1 using the similarity solutions above. What is z_f and what do you think might happen at greater heights?