A Note on Gill-type Solutions to Boundary Layer Quasi-Equilibrium Equations and Discussion Concerning the Weak Temperature Gradient Approximation

Equations linearized about a resting state in radiative-convective equilibrium on an equatorial beta plane, with barotropic mode ignored and first baroclinic mode retained:

$$\frac{\partial u}{\partial t} = (T_s - \overline{T})\frac{\partial s^*}{\partial x} + \beta y v - r u, \tag{1}$$

$$\frac{\partial v}{\partial t} = (T_s - \overline{T})\frac{\partial s^*}{\partial y} - \beta y u - r v, \tag{2}$$

$$\frac{\partial s^*}{\partial t} = \frac{\Gamma_d}{\Gamma_m}\left(\dot{Q}_{rad} + \frac{\partial s_d}{\partial z}(\epsilon_p M' - w)\right), \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{w}{H} = 0, \tag{4}$$

where u, v, w, and s^* are the three velocity components and the saturation moist entropy of the troposphere, respectively, r is a linear damping coefficient acting on the first baroclinic mode, T_s is an average surface temperature, $\overline{T}$ is a mean temperature along a moist adiabat, H is a depth scale for the lower troposphere, $\frac{\Gamma_d}{\Gamma_m}$ is the ratio of the dry and moist adiabatic lapse rates, $\dot{Q}_{rad}$ is the radiative tendency of $ln(\theta)$ multiplied by heat capacity at constant pressure, $\frac{\partial s_d}{\partial z}$ is the background value of the lapse rate of dry air entropy, ϵ_p is a mean precipitation efficiency, and M' is the (perturbation) convective updraft mass flux. The nonlinear form of M, according to boundary layer QE theory, is

$$M = w + \frac{C_k|\mathbf{V}|(s_0^* - s^*)}{(s^* - s_m)}, \tag{5}$$

where C_k is a surface enthalpy exchange coefficient, $|\mathbf{V}|$ is the surface wind speed, s_0^* is the saturation entropy of the sea surface, and s_m is the entropy of the lower to middle troposphere. In (5) we have used the convective neutrality assumption that the subcloud layer entropy equals the saturation entropy of the free troposphere.

Linearizing this about an assumed mean easterly wind gives

$$M' = w' + C_k\left(|\mathbf{V}|'\frac{\overline{(s_0^* - s^*)}}{(s^* - s_m)} + \frac{\overline{|\mathbf{V}|}}{(s^* - s_m)}(s_0^{*\prime} - s^{*\prime}) - \overline{|\mathbf{V}|}\frac{\overline{(s_0^* - s^*)}}{(s^* - s_m)^2}(s^{*\prime} - s_m{}')\right). \tag{6}$$

Here the primes represent perturbation quantities but are henceforth dropped from the notation.

We represent the perturbation radiative heating in (3) as a simple Newtonian relaxation:

$$\dot{Q}_{rad} = -\gamma s^*, \tag{7}$$

and we note that including s_m' would require an equation for the mid-tropospheric entropy. We linearize $|\mathbf{V}|$ about a mean zonal wind $\overline{U}$ as

$$|\mathbf{V}|' = \frac{\overline{U}u'}{\overline{|\mathbf{V}|}}. \tag{8}$$

Our equation set thus consists of (1)-(4) and (6)-(8).

Scalings

First normalize all zonal length scales by the radius of the Earth, a, and define a meridional scale:

$$L_y^4 \equiv \frac{\Gamma_d}{\Gamma_m}(T_s - \overline{T})H\frac{\partial s_d}{\partial z}\frac{1 - \epsilon_p}{\beta^2}.$$

A typical value of L_y is around 1200 km.

We then apply the following scaling of the dependent and independent variables:

$$x \to a\,x$$

$$y \to L_y\,y$$

$$t \to \frac{a}{\beta L_y^2}\,t$$

$$u \to \frac{aC_k\overline{|\mathbf{V}|}}{H}\,u$$

$$v \to \frac{L_yC_k\overline{|\mathbf{V}|}}{H}\,v$$

$$s^* \to \frac{aC_k\overline{|\mathbf{V}|}\beta L_y^2}{H(T_s - \overline{T})}\, s$$

Here, a is the radius of the Earth.

We apply seperate scalings for the ocean temperature perturbations and lower troposphere entropy perturbations:

$$s_0^* \to \frac{1 - \epsilon_p}{\epsilon_p}\overline{(s^* - s_m)}\, s_0$$

$$s_m \to \frac{1 - \epsilon_p}{\epsilon_p} \frac{\overline{(s^* - s_m)}^2}{\overline{(s_0^* - s^*)}}\, s_m$$

We next define nondimensional parameters:

$$\mathcal{R} \equiv \frac{ra}{\beta L_y^2},$$

$$\alpha \equiv \frac{1 - \epsilon_p}{\epsilon_p} \frac{aC_k}{H} \frac{\overline{U}}{\sqrt{\overline{U}^2 + u*^2}} \frac{\overline{(s_0^* - s^*)}}{\overline{(s^* - s_m)}},$$

$$\chi \equiv \frac{a\beta L_y^2}{H(T_s - \overline{T})(1 - \epsilon_p)} \left[\frac{\gamma}{\frac{\partial s_d}{\partial z}} + \frac{\epsilon_p C_k \overline{|\mathbf{V}|}}{\overline{(s^* - s_m)}} \frac{\overline{s_0^* - s_m}}{\overline{s^* - s_m}} \right],$$

and

$$\delta \equiv \left(\frac{a}{L_y} \right)^2.$$

With these scalings, and making use of (6)-(8) and (4), equations (1)-(3) become

$$\frac{\partial u}{\partial t} = \frac{\partial s}{\partial x} + yv - \mathcal{R}u, \tag{9}$$

$$\frac{\partial v}{\partial t} = \delta\left(\frac{\partial s}{\partial y} - yu \right) - \mathcal{R}v, \tag{10}$$

and

$$\frac{\partial s}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \alpha u + s_0 + s_m - \chi s. \tag{11}$$

Note that the Sobel and Bretherton Weak Temperature Gradient approximation consists of dropping the time derivative and α and χ terms in (11).

It is of some interest to look for steady solutions to (9)-(11), which is like a Gill model except that the ocean temperature perturbation (s_0) is specified rather than the heating

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per se. Dropping the momentum damping ($\mathcal{R}$), the s_m term in (11) (which we have not written an equation for) and the time dependence, (9)-(11) may be combined into a single equation for s:

$$\frac{\partial s}{\partial x} + \alpha y \frac{\partial s}{\partial y} - \chi y^2 s = -y^2 s_0. \tag{12}$$

As Sobel and Bretherton have noted, (12) is ill-posed when $\chi = 0$ unless s_0 integrates to zero around the equator. It is not clear whether the WISHE term (multiplied by α here) spares one from this requirement off the equator. Any ideas?

Consider a class of ocean temperature perturbations that are sinusoidal in longitude:

$$s_0 = \text{RE}[G(y)e^{ikx}]. \tag{13}$$

Then, letting

$$s = \text{RE}[J(y)e^{ikx}],$$

the general inhomogeneous solution to (12) is

$$J = y^{-ik/\alpha} e^{\frac{1}{2}\chi y^2/\alpha} \int_0^y Gu^{1+ik/\alpha} e^{-\frac{1}{2}\chi u^2/\alpha} du. \tag{14}$$

This is tough to evaluate numerically for very small α, but we provide a solution to (12) in the case that $\alpha = 0$.

FORTRAN programs for solving (14) and a MATLAB script for graphing the output are available. In the following, I take

$$G(y) = e^{-by^2}$$

Figure 1a shows the temperature (saturation entropy) and horizontal flow, for $\alpha = 0$, $k = 2$, $b = 1.5$ and my best estimate of χ, 1.5. Figure 1b shows s_0 and w for the same solution. For clarity, only a half zonal wavelength is shown.
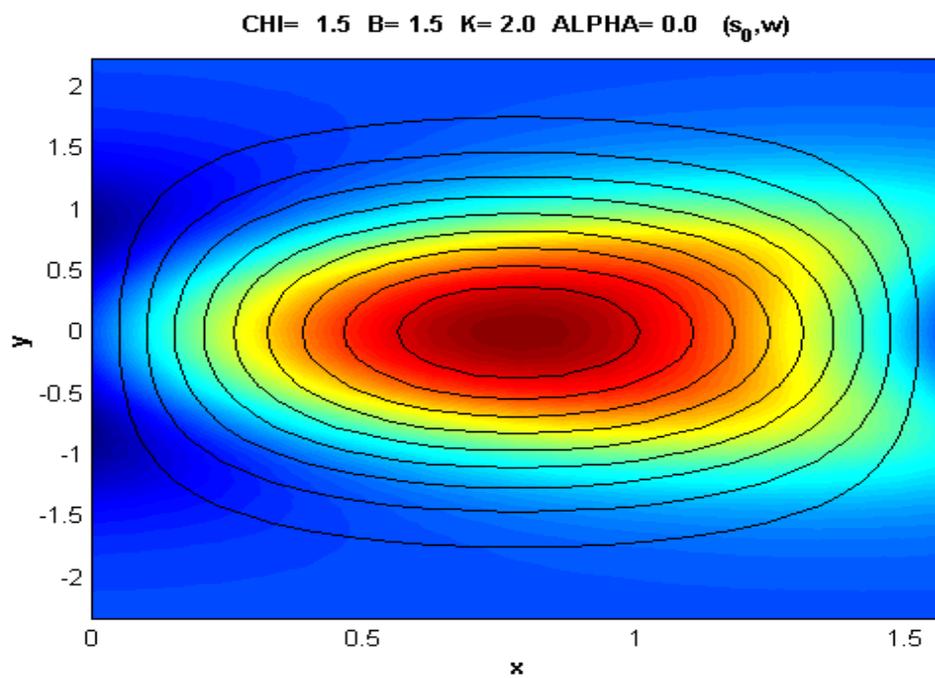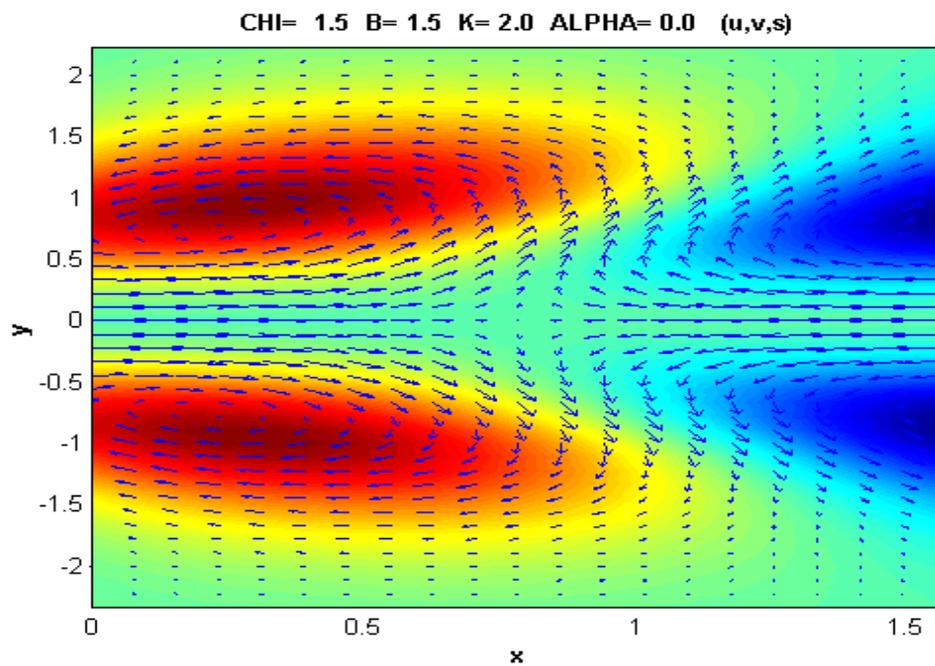
Figures 2 shows the same solutions, but with $\chi = 0$. comparing the two figures, it is seen that the main effect of the WTG approximation is to shift the winds, pressures

and temperatures westward relative to the SST anomaly, as already noted by Sobel and Bretherton. In WTG, the zonal winds and SST are exactly out of phase on the equator; otherwise, with reasonable values of χ, there are westerlies over the maximum SST. In WTG, the westerlies can be phase shifted to the east by including momentum damping, as shown by S& B.

Figure 3 shows solutions for the same parameter values as Figure 1, except that $\alpha = -1$. WISHE has a nontrivial effect on the solutions, phase shifting the zonal wind to be almost in phase with the SST and broadening the disturbance in the meridional direction as well.

The WTG approximation, while it does not change the solutions dramatically, does phase shift them in a way that may cause problems in trying to simulate, e.g., El Nino. Nor am I sure that making WTG simplifies (9)-(11) (whether time dependent or not) in a substantive way. It does, in my opinion, provide a convenient way to think about the response of vertical motion to perturbations in the surface enthalpy flux.

The application of boundary layer QE to this problem does have the advantage that the SST, which is the longest time scale perturbation in the system, appears as the forcing of the Gill model, and not the mass fluxes, though to be sure, they are mostly in phase with each other. It is also clear that WISHE can offset the response from the SST in a potentially substantial way.
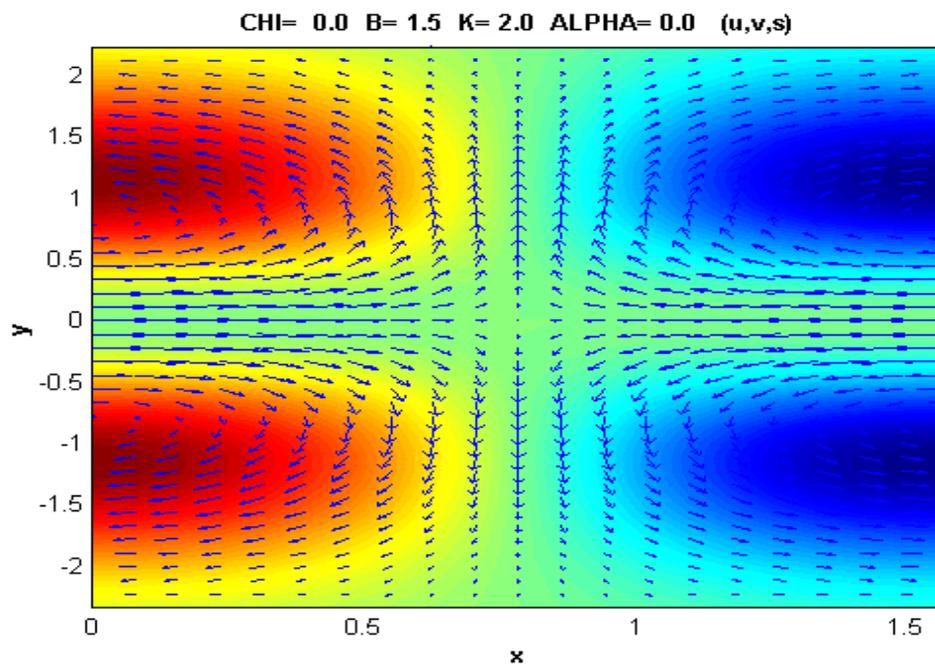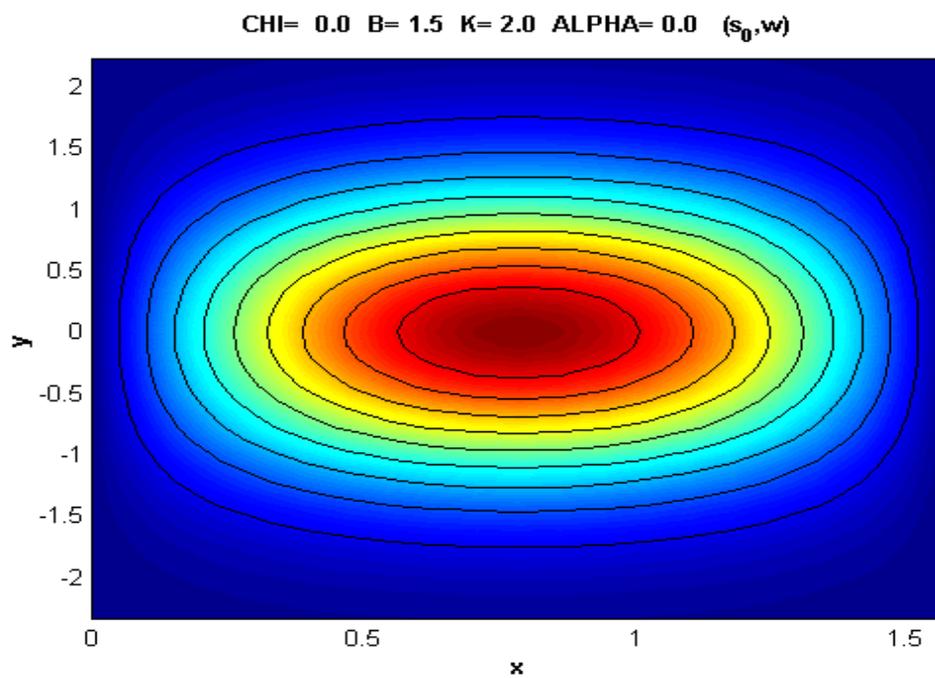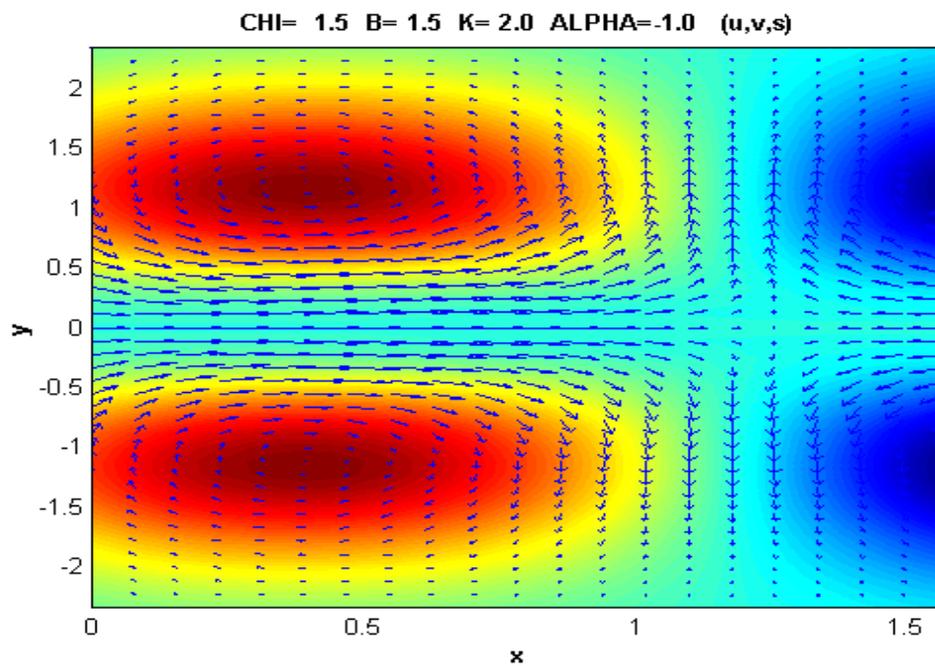
CHI= 1.5 B= 1.5 K= 2.0 ALPHA= 0.0   (u,v,s)

Figure 1a

CHI= 1.5 B= 1.5 K= 2.0 ALPHA= 0.0   (s_0,w)

Figure 1b

CHI= 0.0 B= 1.5 K= 2.0 ALPHA= 0.0 (u,v,s)

Figure 2a

CHI= 0.0 B= 1.5 K= 2.0 ALPHA= 0.0 (s_0,w)

Figure 2b

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CHI= 1.5 B= 1.5 K= 2.0 ALPHA=-1.0 (u,v,s)

Figure 3a

CHI= 1.5 B= 1.5 K= 2.0 ALPHA=-1.0 (s_0,w)

Figure 3b

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