Chapter 8

Internal Gravity Waves: Basics

Supplemental reading:

Holton (1979), sections 7.1–4

Houghton (1977), sections 8.1–3

Pedlosky (2003)

Lindzen (1973)

Atmospheric waves (eddies) are important in their own right as major components of the total circulation. They are also major transporters of energy and momentum. For a medium to propagate a disturbance as a wave there must be a restoring 'force', and in the atmosphere, this arises, primarily, from two sources: conservation of potential temperature in the presence of positive static stability and from the conservation of potential vorticity in the presence of a mean gradient of potential vorticity. The latter leads to what are known as Rossby waves. The former leads to internal gravity waves (and surface gravity waves as well). Internal gravity waves are simpler to understand and clearly manifest the various ways in which waves interact with the mean state. For reasons which will soon become clear, internal gravity waves are not a dominant part of the midlatitude tropospheric circulation (though they are important). Nonetheless, we will study them in some detail – as prototype atmospheric waves. As a bonus, the theory we develop will be sufficient to allow us to understand atmospheric tides, upper atmosphere turbulence, and the quasi-biennial oscillation of the stratosphere. We will also use our results to speculate on the circulation of Venus. The mathematical apparatus needed will, with one exception, not go beyond understanding the simple harmonic oscillator equation. The exception is that I will use elementary WKB theory (without turning points). Try to familiarize yourself with this device, though it will be briefly sketched in Chapter 4.

8.1 Some general remarks on waves

A wave propagating in the x,z-plane will be characterized by functional dependence of the form

$$\cos(\sigma t - kx - \ell z + \phi).$$

We will refer to $k\hat{i}+\ell\hat{k}$ as the wave vector; the period $\tau = 2\pi/\sigma$; the horizontal wavelength $= 2\pi/k$; the vertical wavelength $= 2\pi/\ell$. Phase velocity is given by

$$\frac{\sigma}{k}\hat{i} + \frac{\sigma}{\ell}\hat{k},$$

while group velocity is given by

$$\frac{\partial\sigma}{\partial k}\hat{i} + \frac{\partial\sigma}{\partial\ell}\hat{k}.$$

When phase velocity and group velocity are the same we refer to the wave as non-dispersive; otherwise it is dispersive; that is, different wavelengths will have different phase speeds and a packet will disperse.

8.1.1 Group and signal velocity

The role of the group velocity in this matter is made clear by the following simple argument. Let us restrict ourselves to a signal $f(t, x)$, where

$$f(t, 0) = C(t)e^{i\omega_0 t}.$$

We may Fourier expand $C(t)$:

$$C(t) = \int_{-\infty}^{\infty} B(\omega)e^{i\omega t}d\omega,$$

and then

$$f(t,0) = \int_{-\infty}^{\infty} B(\omega) e^{i(\omega+\omega_0)t} d\omega.$$

Away from $x = 0$

$$f(t,x) = \int_{-\infty}^{\infty} B(\omega) e^{i[(\omega+\omega_0)t - k(\omega+\omega_0)x]} d\omega$$

(N.B. $k(\omega+\omega_0)$ means, in this instance, that k is a function of $\omega+\omega_0$).

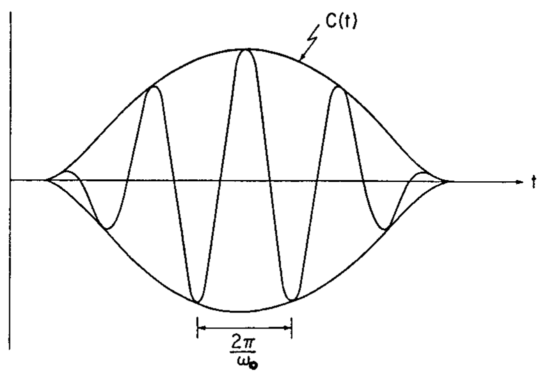

Figure 8.1: Modulated carrier wave.

Now

$$k(\omega+\omega_0) = k(\omega_0) + \left.\frac{dk}{d\omega}\right)_{\omega_0} \omega +$$

and

$$f(t,x) = \int_{-\infty}^{\infty} B(\omega) e^{i(\omega_0 t - k(w_0)x)} e^{i(\omega t - \frac{dk}{d\omega}\omega x + \dots)} d\omega.$$

Assume $B(\omega)$ is sufficiently band-limited so that the first two terms in the Taylor expansion of k are sufficient. Then

$$\begin{aligned} f(t,x) &= e^{i(\omega_0 t - k(\omega_0)x)} \int_{-\infty}^{\infty} B(\omega) e^{i\omega(t - \frac{dk}{d\omega}x)} d\omega \\ &= e^{i(\omega_0 t - k(\omega_0)x)} C\left(t - \frac{dk}{d\omega}x\right). \end{aligned}$$

We observe that the information (contained in C) travels with the group velocity

$$c_G = \left(\frac{dk}{d\omega}\right)^{-1} = \frac{d\omega}{dk}.$$

8.2 Heuristic theory (no rotation)

In studying atmospheric thermodynamics you have seen that a neutrally buoyant blob in a stably stratified fluid will oscillate up and down with a frequency N. Applying this to the configuration in Figure 8.2, we get

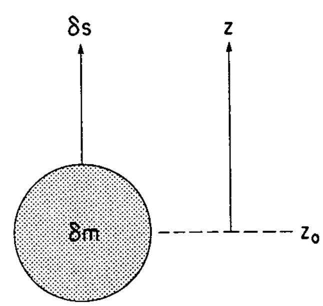

Figure 8.2: Blob in stratified fluid.

$$\frac{d^2 \delta s}{dt^2} = -N^2 \delta s,$$

where

$$N^2 = \underbrace{\frac{g}{T_0}\left(\frac{dT_0}{dz} + \frac{g}{c_p}\right)}_{static\ stability} \quad \left(-\frac{g}{\rho_0}\frac{d\rho_0}{dz} \text{ in a Boussinesq fluid}\right).$$

The restoring force (per unit mass), $F_b = -N^2\delta s$, is directed vertically.

Now consider the following situation where a corrugated sheet is pulled horizontally at a speed c through a stratified fluid (*viz.*, Figure 8.3). Wave

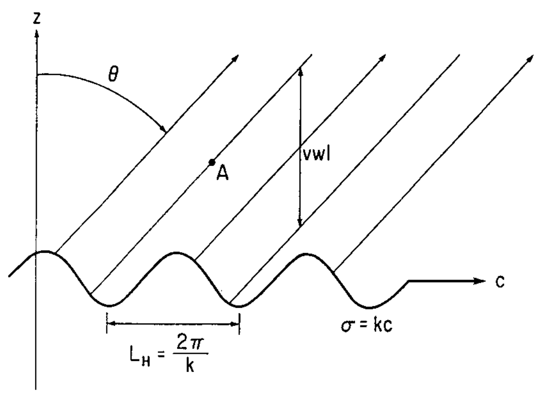

Figure 8.3: Corrugated lower surface moving through a fluid.

motions will be excited in the fluid above the plate with frequency, $\sigma = kc$. For our oscillating blob we used '$F = ma$'. Normally, we cannot use this for fluid flows because of the pressure force. However, in a wave field there will be lines of constant phase. Along these lines the pressure perturbation will be constant and hence pressure gradients along such lines will be zero, and the acceleration of the fluid along such lines will, indeed, be given by $F = ma$.

Assume such lines make an angle Θ with respect to the vertical. The projection of the buoyancy force is then

$$F = -N^2\delta z \cos\Theta.$$

Also

$$\delta z = \delta s \cos\Theta$$

so

$$F = -N^2 \cos^2\Theta\, \delta s$$

and

$$\frac{d^2 \delta s}{dt^2} = -N^2 \cos^2\Theta\, \delta s.$$

Thus

$$\sigma^2 = N^2 \cos^2\Theta = k^2 c^2;$$

that is, kc determines Θ.

Θ is also related to the ratio of horizontal and vertical wavelengths:

$$\begin{aligned} \tan\Theta &= L_H/L_v = \ell/k \\ \tan^2\Theta &= \frac{\ell^2}{k^2} = \frac{1-\cos^2\Theta}{\cos^2\Theta} = \frac{1-\frac{k^2c^2}{N^2}}{\frac{k^2c^2}{N^2}}, \end{aligned}$$

which is, in fact, our dispersion relation. Note that ℓ is the vertical wavenumber, where $\ell = 2\pi/\text{vwl}$.

$$\ell^2 = \left(\frac{N^2}{k^2c^2} - 1\right) k^2 = \left(\frac{N^2}{\sigma^2} - 1\right) k^2. \tag{8.1}$$

We see, immediately, that vertical propagation requires that $\sigma^2 < N^2$. When $\sigma^2 > N^2$, the buoyancy force is inadequate to maintain an oscillation and the perturbation decays with height. Equation 8.1 may be solved for σ:

$$\sigma = \pm \frac{Nk}{(k^2 + \ell^2)^{1/2}} \tag{8.2}$$

Now

$$c_{p_x} = \frac{\sigma}{k} = \pm \frac{N}{(k^2 + \ell^2)^{1/2}} \tag{8.3}$$

$$c_{p_z} = \frac{\sigma}{\ell} = \pm \frac{Nk/\ell}{(k^2 + \ell^2)^{1/2}} \tag{8.4}$$

$$c_{g_x} = \frac{\partial \sigma}{\partial k} = \pm \frac{N\ell^2}{(k^2 + \ell^2)^{3/2}} \tag{8.5}$$

$$c_{g_z} = \frac{\partial \sigma}{\partial \ell} = \pm \frac{-Nk\ell}{(k^2 + \ell^2)^{3/2}}. \tag{8.6}$$

Thus c_{p_x} and c_{g_x} are of the same sign, while c_{p_z} and c_{g_z} are of opposite sign. What this means is easily seen from Figure 8.3. Since the wave is forced by the moving corrugated plate, the motions must angle to the right of the vertical so that the moving corrugation is pushing the fluid ($\overline{p'w'}$ is positive). (Equation 8.2 permits Θ to be positive or negative, but negative values would correspond to the fluid pushing the plate.) But then phase line A, which is moving to the right, is also seen by an observer at a fixed x as moving downward. Thus for an internal gravity wave, upward wave propagation is associated with downward phase propagation[1].

Before moving beyond this heuristic treatment, two points should be mentioned.

1. We have already noted that vertical propagation ceases, for positive k^2, when σ^2 exceeds N^2. (For the atmosphere $2\pi/N \sim 5$ min.) It is also worth noting what happens as $\sigma \to 0$. The vertical wavelength $(2\pi/\ell) \to 0$, as does c_{g_z}. We may anticipate that in this limit any damping will effectively prevent vertical propagation. Why?

2. The excitation of gravity waves by a moving corrugated plate does not seem terribly relevant to either the atmosphere or the ocean. So we should see what is needed more generally. What is needed is anything which will, as seen by an observer moving with the fluid, move height surfaces up and down. Thus, rather than move the corrugations through the fluid, it will suffice for the fluid to move past fixed corrugations – or mountains for that matter. Similarly, a heat source moving relative to the fluid will displace height surfaces and excite gravity waves. The daily variations of solar insolation act this way. Other

[1]This is not always true if the fluid is moving relative to the observer. The reader is urged to examine this possibility.

more subtle excitations of gravity waves arise from fluid instabilities, collapse of fronts, squalls, and so forth.

The above, heuristic, analysis tells us much about gravity waves – and is simpler and more physical than the direct application of the equations. However, it is restricted to vertical wavelengths much shorter than the fluid's scale height; it does not include rotation, friction, or the possibility that the unperturbed basic state might have a spatially variable flow. To extend the heuristic approach to such increasingly complicated situations becomes almost impossible. It is in these circumstances that the equations of motion come into their own.

8.3 Linearization

Implicit in the above was that the gravity waves were small perturbations on the unperturbed basic state. What does this involve in the context of the equations of motion? Consider the equation of x-momentum on a non-rotating plane for an inviscid fluid:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{8.7}$$

In the absence of wave perturbations assume a solution of the following form:

$$\begin{aligned} u &= u_0(y,z) \\ p &= p_0(y,z) \\ \rho &= \rho_0(y,z) \\ v_0 &= 0 \\ w_0 &= 0. \end{aligned}$$

Equation 8.7 is automatically satisfied. Now, add to the basic state, perturbations u', v', w', ρ', p'. Equation 8.7 becomes

$$\begin{aligned} \frac{\partial u'}{\partial t} + u_0\frac{\partial u'}{\partial x} &+ u'\frac{\partial u'}{\partial x} + v'\frac{\partial u_0}{\partial y} + v'\frac{\partial u'}{\partial y} \\ &+ w'\frac{\partial u_0}{\partial z} + w'\frac{\partial u'}{\partial z} = -\frac{1}{(\rho_0+\rho')}\frac{\partial p'}{\partial x}. \end{aligned} \tag{8.8}$$

Terms involving the basic state alone cancelled since the basic state must, itself, be a solution. Linearization is possible when the perturbation is so small that terms quadratic in the perturbation are much smaller than terms linear in the perturbation. The linearization of Equation 8.8 is

$$\frac{\partial u'}{\partial t} + u_0\frac{\partial u'}{\partial x} + v'\frac{\partial u_0}{\partial y} + w'\frac{\partial u_0}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}. \tag{8.9}$$

It is sometimes stated that linearization requires $|u'| \ll |u_0|$, but clearly this is too restrictive. The precise assessment of the validity of linearization depends on the particular problem being solved. For our treatment of gravity waves we will always assume the waves to be linearizable perturbations.

Before proceeding to explicit solutions we will prove a pair of theorems which are at the heart of wave–mean flow interactions. Although we will not be using these theorems immediately, I would like to present them early so that there will be time to think about them. I will also discuss the relation between energy flux and the direction of energy propagation.

8.4 Eliassen-Palm theorems

Let us assume that rotation may be ignored. Let us also ignore viscosity and thermal conductivity. Finally, let us restrict ourselves to basic flows where $v_0 = w_0 = 0$ and $u_0 = u_0(z)$. Also, let us include thermal forcing of the form

$$J = \tilde{J}(z)e^{ik(x-ct)}$$

and seek solutions with the same x and t dependence. The equation for x-momentum becomes

$$\rho_0\left(\frac{\partial u'}{\partial t} + u_0\frac{\partial u'}{\partial x} + w'\frac{\partial u_0}{\partial z}\right) + \frac{\partial p'}{\partial x} = 0$$

or

$$\rho_0(u_0 - c)\frac{\partial u'}{\partial x} + \rho_0 w' \frac{du_0}{dz} + \frac{\partial p'}{\partial x} = 0. \tag{8.10}$$

Similarly for w',

$$\rho_0(u_0 - c)\frac{\partial w'}{\partial x} + \rho' g + \frac{\partial p'}{\partial z} = 0. \tag{8.11}$$

Continuity yields

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} + \frac{1}{\rho_0}\left((u_0 - c)\frac{\partial \rho'}{\partial x} + w' \frac{d\rho_0}{dz}\right) = 0. \tag{8.12}$$

From the energy equation (expressed in terms of p and ρ instead of T and ρ; do the transformations yourself)

$$(u_0 - c)\frac{\partial p'}{\partial x} + w' \frac{dp_0}{dz} = \gamma g H \left((u_0 - c)\frac{\partial \rho'}{\partial x} + w' \frac{d\rho_0}{dz}\right) + (\gamma - 1)\rho_0 J. \tag{8.13}$$

(Remember that $\gamma = c_p/c_v$. Also, $H = RT/g$.)
Now multiply Equation 8.10 by $(\rho_0(u_0 - c)u' + p')$ and average over x:

$$\begin{aligned}(\rho_0(u_0 - c)u' + p') \quad &\cdot \quad \frac{\partial}{\partial x}(\rho_0(u_0 - c)u' + p') \\ &+ \quad \rho_0 \frac{du_0}{dz}(\rho_0(u_0 - c)u'w' + p'w') = 0.\end{aligned}$$

The first term vanishes when averaged over a wavelength in the x-direction while the second term yields

$$\overline{p'w'} = -\rho_0(u_0 - c)\overline{u'w'}. \tag{8.14}$$

Equation 8.14 is Eliassen and Palm's first theorem. $\overline{p'w'}$ is the vertical energy flux associated with the wave (this is not quite true – but its sign is the sign of wave propagation; we will discuss this further later in this chapter), while $\rho_o\overline{u'w'}$ is the vertical flux of momentum carried by the wave (Reynold's stress). This theorem tells us that the momentum flux is such that if deposited in the mean flow it will bring u_0 towards c. Stated differently, an upward propagating wave carries westerly momentum if $c > u_0$ and easterly momentum if $c < u_0$!

Their second theorem, which tells how $\rho_0\overline{u'w'}$ varies with height, is harder to prove.

Let

$$ik(u_0 - c)\zeta = w = (u_0 - c)\frac{\partial \zeta}{\partial x} \tag{8.15}$$

(this simply defines the vertical displacement, ζ, for small perturbations). Substituting Equation 8.15 into Equation 8.13 yields

$$\frac{p'}{\gamma g H} + \frac{\rho_0}{g}N^2\zeta = \rho' + \frac{(\gamma - 1)}{\gamma g H}\frac{\rho_0 J}{ik(u_0 - c)}. \tag{8.16}$$

Substituting Equation 8.16 into Equation 8.11 yields

$$\rho_0(u_0 - c)\frac{\partial w'}{\partial x} + \frac{p'}{\gamma H} + \rho_\circ N^2\zeta - \frac{(\gamma - 1)}{\gamma H}\frac{\rho_0 J}{ik(u_0 - c)} + \frac{\partial p'}{\partial z} = 0. \tag{8.17}$$

Substituting Equation 8.13 into Equation 8.12 yields

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} + \frac{1}{\rho_0 \gamma g H}(u_0 - c)\frac{\partial p'}{\partial x} - \frac{1}{\gamma H}w' - \frac{(\gamma - 1)}{\gamma g H}J = 0. \tag{8.18}$$

Now multiply Equation 8.10 by u', Equation 8.17 by w', and Equation 8.18 by p' and sum them. Equation 8.10 multiplied by u' yields

$$\frac{\partial}{\partial x}\left(\frac{\rho_0(u_0 - c)u'^2}{2}\right) + \rho_0\frac{du_0}{dz}u'w' + u'\frac{\partial p}{\partial x} = 0. \tag{8.19}$$

Equation 8.17 multiplied by w' yields

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\rho_0(u_0 - c)w'^2}{2}\right) \quad + \quad \frac{w'p'}{\gamma H} + \frac{\partial}{\partial x}\left(\frac{\rho_0 N^2(u_0 - c)\zeta^2}{2}\right)& \\ - \quad \frac{(\gamma - 1)\rho_0\zeta J}{\gamma H} + w'\frac{\partial p'}{\partial z} = 0,&\end{aligned} \tag{8.20}$$

where Equation 8.15 has been used to define ζ.
Equation 8.18 multiplied by p' yields

$$p'\frac{\partial u'}{\partial x} + p'\frac{\partial w'}{\partial z} + \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{(u_0 - c)}{\rho_0 \gamma g H}p'^2\right) - \frac{p'w'}{\gamma H} - \frac{\kappa}{gH}p'J = 0. \tag{8.21}$$

Adding Equations 8.19, 8.20, and 8.21 yields

$$\begin{aligned}\frac{\partial}{\partial x}\left\{\frac{1}{2}\rho_0(u_0 - c)u'^2\right. &+ \frac{1}{2}\rho_0(u_0 - c)w'^2 + \frac{1}{2}\rho_0 N^2(u_0 - c)\zeta^2 \\ &+ \left.\frac{1}{2}\frac{(u_0 - c)}{\rho_0\gamma g H}p'^2 + p'u'\right\} \\ +\frac{\partial}{\partial z}(p'w') + \rho_0\frac{du_0}{dz}u'w' &- \frac{\kappa\rho_0\zeta J}{H} - \frac{\kappa}{gH}p'J = 0\end{aligned} \tag{8.22}$$

(where $\kappa = \frac{\gamma - 1}{\gamma}$).

The last two terms can be rewritten

$$-\kappa\rho_0\left(\frac{\zeta}{H} + \frac{p'}{p_0}\right)J.$$

Averaging with respect to x (assuming periodicity):

$$\frac{\partial}{\partial z}(\overline{w'p'}) = -\frac{d\bar{u}}{dz}\rho_0\overline{u'w'} + \kappa\rho_0\overline{\left(\frac{\zeta}{H} + \frac{p'}{p_0}\right)J}. \tag{8.23}$$

Finally, we substitute Equation 8.14 in Equation 8.23 to obtain one version of Eliassen and Palm's second theorem:

$$\frac{\partial}{\partial z}(\rho_0\overline{u'w'}) = -\frac{\kappa\rho_0}{(u_0 - c)}\overline{DJ}, \tag{8.24}$$

where

$$D = \frac{\zeta}{H} + \frac{p'}{p_0}.$$

Eliassen and Palm (1961) developed their theorem for the case where $J = 0$. In that case

$$\frac{\partial}{\partial z}(\rho_0 \overline{u'w'}) = 0. \tag{8.25}$$

Notice that theorem 1 (Equation 8.14) tells us that the sign of $\rho_0 \overline{u'w'}$ is independent of $\frac{du_0}{dz}$, while theorem 2 (Equation 8.25) tells us that in the absence of

(i) damping,

(ii) local thermal forcing, and

(iii) critical levels (where $u_0 - c = 0$)

no momentum flux is deposited or extracted from the basic flow. Contrast Equations 8.14 and 8.25 with what one gets assuming Reynold's stresses are due to locally generated turbulence (where eddy diffusion is down-gradient), and consider the fact that in most of the atmosphere, eddies are, in fact, waves.

From Equation 8.25 and Equation 8.14 we also see that the quantity $\overline{p'w'}/(u_0 - c)$ and not $\overline{p'w'}$ is conserved. The former is sometimes referred to as wave action.

8.4.1 'Moving flame effect' and the super-rotation of Venus' atmosphere

The role of the right hand side of Equation 8.24 is of some interest. It is clear that $\overline{DJ} \neq 0$, at least in a situation which allows the radiation of waves to infinity. Consider thermal forcing in some layer (as shown in Figure 8.4). Above J there will be a momentum flux which must come from some place and cannot come from the region below J; hence, it must come from the thermal forcing region. Moreover, the flux divergence within the heating region must be such as to accelerate the fluid within the heating region in a direction opposite to c. This mechanism has sometimes been called the 'moving flame effect,' and has been suggested as the mechanism responsible for maintaining an observed 100 m/s zonal flow in Venusian cloud layer, the thermal forcing being due to the absorption of sunlight at the cloud top. Since this flow is in the direction of Venus' rotation, it is referred to as 'super-rotation'.

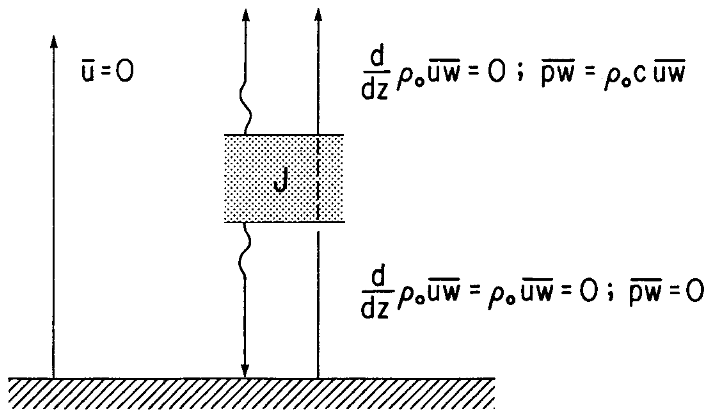

Figure 8.4: Fluxes associated with a layer of thermal forcing.

8.5 Energy flux

We have already noted that $\overline{p'w'}$ is not the complete expression for the wave flux of energy. It is merely the contribution of the pressure-work term to the flux. In addition, there is the advection by the wave fields of the kinetic energy of the mean flow, $\rho_0\overline{u'w'}U$ (Show this.). Thus, the full expression for the energy flux is

$$F_E = \overline{p'w'} + \rho_0\overline{u'w'}U. \tag{8.26}$$

Note that now $dF_E/dz = 0$ under the conditions for which non-interaction holds. On the other hand, F_E is arbitrary up to a Galilean transformation, and hence may no longer be associated with the direction of wave propagation.

The above difficulty does not depend on U having shear, so we will restrict ourselves to the case where $dU/dz = 0$. We will also ignore time variations in ρ_0 – which is acceptable for a Boussinesq fluid. Finally, we will assume the presence of a slight amount of damping which will lead to some absorption of wave fluxes and the consequent modification of the basic state:

$$\rho_0\frac{\partial U}{\partial t} = -\frac{d}{dz}\rho_0\overline{u'w'} \tag{8.27}$$

and

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{U^2}{2} + Q\right) = -\frac{d}{dz}(\overline{p'w'} + \rho_0 \overline{u'w'}U), \tag{8.28}$$

where Q is a measure of the fluid's heat or internal energy.

Now multiply Equation 8.27 by U,

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{U^2}{2}\right) = -U\frac{d}{dz}\rho_0 \overline{u'w'}, \tag{8.29}$$

and subtract Equation 8.29 from Equation 8.28,

$$\rho_0 \frac{\partial}{\partial t} Q = -\frac{d}{dz}\overline{p'w'}. \tag{8.30}$$

From Equation 8.29, we see that the second term in Equation 8.26 is associated with the alteration of the kinetic energy of the basic state (associated with wave absorption), and that the choice of a Galilean frame can lead to either a decrease or an increase of the mean kinetic energy (Why?). From Equation 8.30 we see that the convergence of $\overline{p'w'}$ is, on the other hand, associated with mean heating. Since we do not wish the absorption of a wave to cool a fluid, we are forced to identify the direction of $\overline{p'w'}$ with the direction of wave propagation. (Recall that the wave is attenuated in the direction of propagation.)

8.6 A remark about 'eddies'

As we proceed with our discussion of eddies, it will become easy to lose track of where we are going. Recall this course has a number of aims:

(i) to familiarize you with the foundations and methodologies of dynamics;

(ii) to use this tool to account for some of the observed motion systems of the atmosphere and oceans; and

(iii) to examine the roles of these systems in the 'general circulation'.

You can be reasonably assured that our approach to (ii) will *not* be very systematic. Even with item (iii), it will not be easy to provide a straightforward treatment – at least partly because a complete answer is not yet available!

In our treatment of the symmetric circulation our approach was to calculate a symmetric circulation and see to what extent it accounted for observations. Our assumption was that the degree to which symmetric models failed pointed the way to the role of eddies. We will follow a similarly indirect path in studying eddies and their rôles. The difficulty here is that there are many kinds of eddies! Broadly speaking, we have gravity and Rossby waves – but these may be forced, or they may be free 'drum head' oscillations, or they may even be instabilities of the basic flow. We shall not, of course, study these various eddies at random. We investigate various waves because they seem suitable to particular phenomena. However, there always remains a strong element of 'seeing what happens' which should neither be forgotten nor underestimated.

8.7 Formal mathematical treatment

We shall now return to the problem considered heuristically earlier in this chapter in order to deal with it in a more formal manner. This approach will confirm and extend the results obtained heuristically. The new results are important in their own right; they will also permit us to introduce terminology and concepts which are essential to the discussion of atmospheric tides in Chapter 9. Our equations will be the Boussinesq equations for perturbations on a static basic state:

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} \tag{8.31}$$

$$\rho_0 \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} - g\rho' \tag{8.32}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{8.33}$$

$$\frac{\partial \rho'}{\partial t} + w' \frac{d\rho_0}{dz} = 0, \tag{8.34}$$

where ρ_0 and $\frac{d\rho_0}{dz}$ are treated as constants.
We seek solutions of the form $e^{i(kx-\sigma t)}$:

$$-\rho_0 i\sigma u + ikp = 0 \tag{8.35}$$

$$-\rho_0 i\sigma w + p_z + g\rho = 0 \tag{8.36}$$

$$-i\sigma\rho + w\rho_{0z} = 0 \tag{8.37}$$

$$iku + w_z = 0. \tag{8.38}$$

(For convenience we have dropped primes on perturbation quantities.) Equations 8.35 and 8.38 imply

$$ik\left(\frac{k}{\sigma}\frac{p}{\rho_0}\right) + w_z = 0. \tag{8.39}$$

Equations 8.36 and 8.37 imply

$$-\rho_0 i\sigma w + p_z + g\left(\frac{\rho_{0z}}{i\sigma}w\right) = 0, \tag{8.40}$$

and eliminating p between Equations 8.39 and 8.40 implies

$$w_{zz} + \left\{\left(-\frac{g\rho_{0z}}{\sigma^2\rho_0} - 1\right)k^2\right\}w = 0, \tag{8.41}$$

or

$$w_{zz} + \left\{\left(\frac{N^2}{\sigma^2} - 1\right)k^2\right\}w = 0. \tag{8.42}$$

Note that if Equation 8.36 is replaced by the hydrostatic relation

$$p_z + g\rho = 0, \tag{8.43}$$

then Equation 8.42 becomes

$$w_{zz} + \left\{\frac{N^2k^2}{\sigma^2}\right\}w = 0;$$

that is, hydrostaticity is okay if $N^2/\sigma^2 \gg 1$. This is essentially the same result we obtained in Chapter 6 by means of scaling arguments. If we assume solutions of the form $e^{i\ell z}$, Equation 8.42 is identical to our heuristic result.

Equation 8.41 allows us to examine the relation between internal and surface waves. Note first, that Equation 8.42 has no homogeneous solution for an unbounded fluid that satisfies either the radiation condition or boundedness as $z \to \infty$ if we assume a rigid boundary at $z = 0$. (Such homogeneous solutions will, however, be possible when we allow such features as compressibility and height variable basic stratification.) For Equation 8.42 to have homogeneous solutions (free oscillations) there must be a bounding upper surface. If this is a free surface (such as the ocean surface; the atmosphere doesn't have such a surface), then the appropriate upper boundary condition is $dp/dt = 0$. Linearizing this condition yields

$$\frac{\partial p'}{\partial t} + w'\frac{dp_0}{dz} = \frac{\partial p'}{\partial t} - w' g\rho_0 = 0 \quad \text{at z = H.} \tag{8.44}$$

Using Equations 8.35 and 8.38, Equation 8.44 becomes

$$w_z = g\frac{k^2}{\sigma^2}w \text{ at z = H.} \tag{8.45}$$

Solving Equation 8.42 subject to Equation 8.45 and the condition

$$w = 0 \text{ at z} = 0 \tag{8.46}$$

leads to our free oscillations. Equation 8.42 will have solutions of the form

$$w = \sinh \mu z,$$

where

$$\mu^2 = \left(1 - \frac{N^2}{\sigma^2}\right)k^2, \tag{8.47}$$

or

$$w = \sin \lambda z,$$

where

$$\lambda^2 = \left(\frac{N^2}{\sigma^2} - 1\right)k^2, \tag{8.48}$$

depending on whether σ^2 is greater or less than N^2. Inserting Equations 8.47 and 8.48 into Equation 8.45 yields

$$\tanh \mu H = \frac{\mu}{g}\frac{\sigma^2}{k^2} \tag{8.49}$$

and

$$\tan \lambda H = \frac{\lambda}{g}\frac{\sigma^2}{k^2}, \tag{8.50}$$

respectively.

8.7.1 Shallow water limit and internal modes

In the limit of a shallow fluid ($\mu H \ll 1$ or $\lambda H \ll 1$) both Equation 8.49 and Equation 8.50 reduce to

$$\frac{\sigma^2}{k^2} = gH, \tag{8.51}$$

which is independent of N^2. (What do the solutions look like?) In addition, Equation 8.50 has an infinite number of solutions. In order to examine the nature of these solutions it suffices to replace Equation 8.45 with

$$w = 0 \quad \text{at } z = H. \tag{8.52}$$

Then our solutions become

$$w = \sin \lambda z,$$

where

$$\lambda = \left(\frac{N^2}{\sigma^2} - 1\right)^{1/2} k = \frac{n\pi}{H},\ n = 1, 2, \ldots\ . \tag{8.53}$$

For simplicity let us use the hydrostatic approximation. Then Equation 8.53 becomes

$$\lambda = \frac{N}{\sigma}k = \frac{n\pi}{H}. \tag{8.54}$$

8.7.2 Equivalent depth

Solving Equation 8.54 for σ^2/k^2 we get

$$\frac{\sigma^2}{k^2} = \frac{N^2 H^2}{n^2 \pi^2} \equiv gh. \tag{8.55}$$

By analogy with Equation 8.51, h is referred to as the equivalent depth of the internal mode (or free oscillation). Similarly, in a forced problem, where we impose σ and k, the relation

$$\frac{\sigma^2}{k^2} = gh$$

defines an equivalent depth for the forced mode. (What happens in a channel where $v = 0$ at $y = 0, L$?) This is not, in general, the equivalent depth of any particular free oscillation (if it is we have resonance); rather it is a measure of the vertical wavenumber or the index of refraction. These terms (generalized to a rotating, spherical atmosphere) play a very important role in the treatment of atmospheric tides. The formalism of equivalent depth will also facilitate our discussion of internal Rossby waves (Chapter 11). Finally, it also plays an important role in the contemporary study of equatorial waves.

8.8 Numerical algorithm

In this chapter, as well as in several subsequent chapters, equations of the form

$$w_{zz} + Q^2 w = J(z), \tag{8.56}$$

will describe the vertical structure of waves. While for most of the examples in this chapter, Q was independent of z, in general $Q = Q(U(z), T(z), \vec{k}, \omega)$, where U is a basic profile of zonal wind, T is a basic profile of temperature, $\vec{k}$ is a wave vector, and ω is a frequency. For the more general problems, it is useful to have a simple way of specifying U and T.

8.8.1 Specifying Basic States

The traditional way of specifying U and T was to approximate the distributions by regions with constant gradients and match solutions across the discontinuities in slope. A much cleaner approach is to represent slopes as follows:

$$\frac{dT}{dz} = T_{z,0} + \sum_{1}^{n} (T_{z,n} - T_{z,n-1}) \left(\frac{\tanh(\frac{z-z_n}{\delta T_n}) + 1}{2} \right), \tag{8.57}$$

where $T_{z,n}$ is the characteristic temperature gradient between z_n and z_{n+1}, and δT_n is the transition width at z_n.

$$\frac{dU}{dz} = U_{z,0} + \sum_{1}^{n} (U_{z,n} - U_{z,n-1}) \left(\frac{\tanh(\frac{z-z_n}{\delta U_n}) + 1}{2} \right), \tag{8.58}$$

where $U_{z,n}$ is the characteristic shear between z_n and z_{n+1}, and δU_n is the transition width at z_n. Equations 8.57 and 8.58 are readily integrated to obtain $T(z)$ and $U(z)$. (Hint: $\ln\left(2\cosh\left(\frac{z-z_i}{\delta_i}\right)\right) \rightarrow \frac{z-z_i}{\delta_i}\text{for}\frac{z-z_i}{\delta_i} > 1, \text{and} \rightarrow \frac{z_i-z}{\delta_i}\text{for}\frac{z-z_i}{\delta_i} < -1.$)

8.8.2 Finite Difference Approximations

We will solve 8.56 numerically by finite difference methods. The grid is specified as follows:

$$z_k = k\Delta, k = 1, \ldots, K+1, \tag{8.59}$$

where the mesh size Δ is given by

$$\Delta = \frac{z^*_{top}}{K+1}. \tag{8.60}$$

We approximate the second z-derivative by the standard formula

$$w_{zz} \approx \frac{w_{k+1} - 2w_k + w_{k-1}}{\Delta^2}, \tag{8.61}$$

where a grid notation has been introduced: $w_k = w(z_k)$. The finite difference version of Equation 8.56 is then

$$w_{k+1} + (\Delta^2 Q_k^2 - 2)w_k + w_{k-1} = \Delta^2 J_k$$

for $k = 1, \ldots, K$. or

$$w_{k+1} + a_k w_k + w_{k-1} = b_k, \tag{8.62}$$

where $a_k \equiv \Delta^2 {Q_k}^2 - 2$, and $b_k \equiv \Delta^2 J_k$.

At $z = 0$ we separately consider boundary conditions of the following forms

$$\frac{dw}{dz} + a_B w = b_B, \tag{8.63}$$

or

$$w = d_B. \tag{8.64}$$

For lower boundary conditions of the form 8.63, the most accurate approach is to introduce a fictitious point, z_{-1}. Then the finite difference form of the lower boundary condition is [2]

$$\frac{w_1 - w_{-1}}{2\Delta} + a_B w_0 = b_B.$$

Applying 8.62 at the lower boundary, and using the boundary condition to eliminate w_{-1}, we then get

$$w_1 + (\frac{a_0}{2} + \Delta a_B) w_0 = \frac{b_0}{2} + \Delta b_B. \tag{8.65}$$

For boundary conditions of the form of 8.64, we combine this boundary condition with Equation 8.62 applied at level 1 to obtain

$$w_2 + a_1 w_1 = b_1 - d_B. \tag{8.66}$$

For our upper boundary condition, we have the same two choices:

$$\frac{dw}{dz} + a_T w = b_T, \tag{8.67}$$

or

$$w = d_T. \tag{8.68}$$

For the upper boundary condition given by Equation 8.67, we have

$$\frac{w_{K+1} - w_{K-1}}{2\Delta} + a_T w_K = b_T.$$

[2]The point is simply that centered differences are accurate to order Δ^2 while one-sided differences are accurate to only order Δ. This can actually represent a very substantial loss of accuracy.

When this is combined with Equation 8.62 evaluated at z_K, we get

$$w_{K-1} + (\frac{a_K}{2} - \Delta a_T)w_K = \frac{b_K}{2} - \Delta b_T. \tag{8.69}$$

For the upper boundary condition given by Equation 8.68, we have

$$w_{K+1} = d_T, \tag{8.70}$$

which is already in the form we will need. Our system of equations now consist in Equation 8.62 subject to a lower boundary condition given by either Equation 8.65 or Equation 8.66, and subject to an upper boundary condition given by either Equation 8.69 or Equation 8.70.

8.8.3 Numerical Algorithm

The above system is easily solved using the up-down sweep method (which is simply Gaussian elimination). To begin, we introduce two new vectors, α_k and β_k, related as follows:

$$w_k = \frac{\beta_k - w_{k+1}}{\alpha_k} \tag{8.71}$$

The up-sweep portion of the algorithm consists in determining α_k and β_k. To do this, we substitute 8.71 into 8.62.

$$w_{k+1} + a_k w_k + \frac{\beta_{k-1} - w_k}{\alpha_{k-1}} = b_k,$$

or

$$w_{k+1} + (a_k - \frac{1}{\alpha_{k-1}})w_k = b_k - \frac{\beta_{k-1}}{\alpha_{k-1}}. \tag{8.72}$$

Rewriting 8.71 as follows

$$w_{k+1} + \alpha_k w_k = \beta_k, \tag{8.73}$$

and comparing 8.72 with 8.73, we immediately get

$$\alpha_k = a_k - \frac{1}{\alpha_{k-1}} \qquad 8.74a$$

$$\beta_k = b_k - \frac{\beta_{k-1}}{\alpha_{k-1}}. \qquad 8.74b$$

Our lower boundary condition can be used to determine either α_0 and β_0, or α_1 and β_1. In either case, 8.74 can be used to determine all subsequent α_k's and β_k's (upward sweep). 8.71 will now give us all w_k's if we have w_K or $w_K + 1$. We get this from the upper boundary condition (either one as appropriate). The above algorithm solves for the complex values of w_k at each level. In practice, it is more convenient to look at the amplitudes and phases where

$$amplitude(w_k) = \left((\mathrm{Re}(w_k))^2 + (\mathrm{Im}(w_k))^2\right)^{1/2}$$

and

$$phase(w_k) = \arctan\left[\frac{Im(w_k)}{Re(w_k)}\right].$$

8.8.4 Testing the Algorithm

The simplicity of Equation 8.56, especially when Q^2 is constant, makes it easy to compare our numerical solutions with analytic solutions under a variety of conditions. This is a proper, but incomplete, test. It is proper because it provides a standard of comparison external to the numerical model itself. It is incomplete, because the case of Q^2 being constant is very special, while for more general variations in Q^2, it becomes much more difficult to obtain analytic solutions. Tests without proper standards of comparison (for example, model intercomparisons) are, in general, lacking in credibility.