Chapter 7

Symmetric circulation models

Supplemental reading:

Lorenz (1967)

Held and Hou (1980)

Schneider and Lindzen (1977)

Schneider (1977)

Lindzen and Hou (1988)

Walker and Schneider (2004)

As we noticed in our perusal of the data, atmospheric fields are far from being zonally symmetric. Some of the deviation from symmetry is forced by the inhomogeneity of the earth's surface, and some is autonomous (travelling cyclones, for example). The two are related; for example, the storm paths along which travelling cyclones travel are significantly determined by the planetary scale waves forced by inhomogeneities in the earth's surface. Nevertheless, the zonally averaged circulation has, over the centuries, been the object of special attention. Indeed, the term 'general circulation' is frequently taken to mean the zonally averaged behavior. This is the viewpoint of Lorenz (1967).

You are urged to read chapters 1,3, and 4 of Lorenz. Chapter 1 is a short and especially insightful discussion of the methodology of studying the atmosphere. As is generally the case in this field, there will be views in

Lorenz which are not universally agreed on, but this hardly diminishes its value.

There are several reasons for focussing on the zonally averaged circulation:

1. Significant motion systems like the tropical tradewinds are well described by zonal averages.

2. The circulation of the atmosphere is only a small perturbation on a rigidly rotating basic state which is zonally symmetric.

3. The zonally averaged circulation is a convenient subset of the total circulation.

Our approach in this chapter will be to inquire how the atmosphere would behave in the absence of eddies. It is hoped that a comparison of such results with observations will lend some insight into what maintains the observed zonally averaged state. In particular, discrepancies may point to the rôle of eddies in maintaining the zonal average. This has been a matter of active controversy to the present.

7.1 Historical review

A very complete historical treatment of this subject is given in chapter 4 of Lorenz. We will only present a limited sketch here. The first treatment of contemporary relevance was that of Hadley (1735). Hadley's aim was to explain the easterly (actually northeasterly in the Northern Hemisphere) tradewinds of the tropics and the prevailing westerlies of middle latitudes. His brief explanation is summarized in Figure 7.1. Ignoring Hadley's error in assuming conservation of velocity rather than angular momentum, Hadley's argument ran roughly as follows:

1. Warm air rises at the equator and flows poleward at upper levels approximately conserving angular momentum. (In view of the remarks at the end of Chapter 6, it is not, however, at all obvious why one would have a meridional circulaton at all. We will discuss this later.) Because the distance from the axis of rotation diminishes with increasing latitude, large westerly currents are produced at high latitudes.

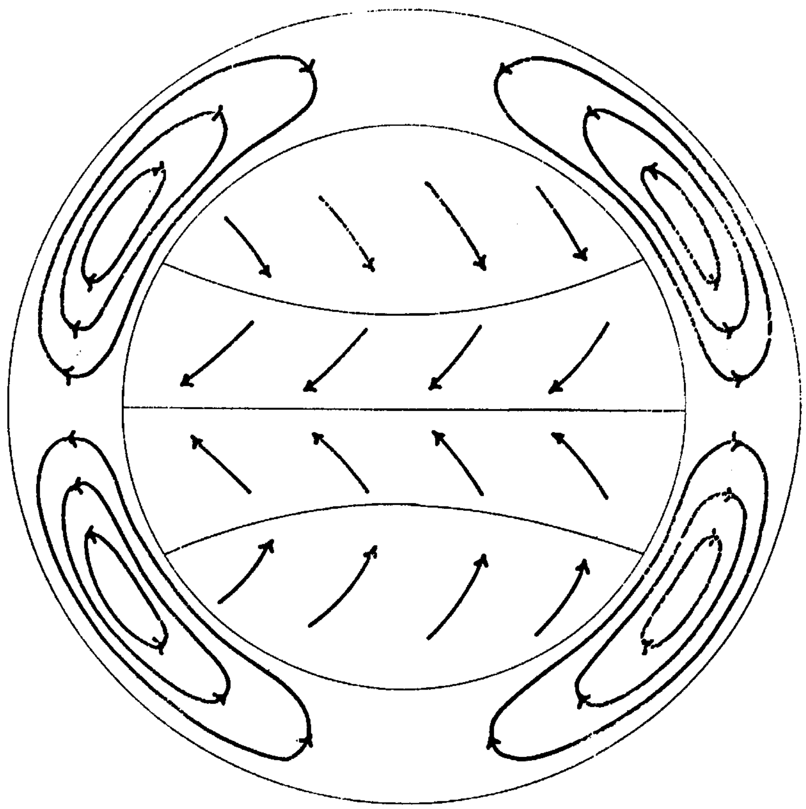


Figure 7.1: A schematic representation of the general circulation of the atmosphere as envisioned by Hadley (1735).

2. The westerly currents would be far larger than observed. It is, therefore, presumed that friction would reduce westerly currents. As a result, the return flow at the surface will have a momentum deficit leading to tropical easterlies.

(In what sense does Hadley's argument constitute an 'explanation'?)

The above model was generally accepted for over a century. The main criticism of this model was that it predicted northwesterly winds at midlatitudes whereas nineteenth century data suggested southwesterly winds (in the Northern Hemisphere). This difficulty was answered independently by Ferrel (1856) and Thomson (1857). Their hypothesized solution is shown in Figure 7.2. Briefly, Ferrel and Thomson supplemented Hadley's

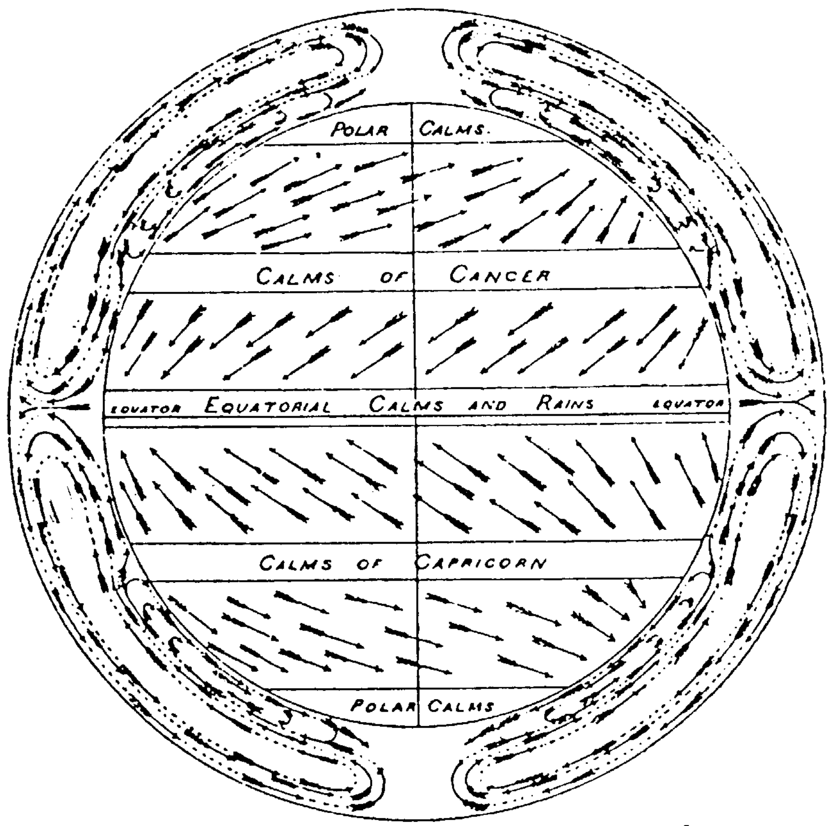


Figure 7.2: The general circulation of the atmosphere according to Thomson (1857).

arguments as follows. They noted that at the latitude at which zonal flow is zero, there must be a maximum in pressure (Why?), and that within the frictional layer next to the surface, a shallow flow will be established down pressure gradients leading to the reversed cell shown in Figure 7.2 (today referred to as a Ferrel cell). *By allowing the Hadley circulation to remain at upper levels, the tropics can continue to supply midlatitudes with angular momentum* which presumably is communicated to the Ferrel cell by friction.

Although there was a general acceptance of the Ferrel-Thomson model of the general circulation in the late nineteenth century, there was also a general uneasiness due to the obvious fact that the observed circulation was not zonally symmetric. Moreover, all the models we have discussed were developed in only a qualitative, verbal way. This is, of course, not surprising since the quantitative knowledge of atmospheric heating, turbulent transfer, and so forth, was almost completely lacking. So was virtually any information about the atmosphere above the surface – except insofar as cloud motions indicated upper level winds.

In 1926, Jeffreys put forth an interesting and influential criticism of all symmetric models as an explanation of midlatitude surface westerlies. He began with an equation for zonal momentum (*viz.* Equation 6.18):

$$\begin{aligned} \frac{\partial u}{\partial t} \quad &+ \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{uv\tan\phi}{a} + \frac{uw}{a} \\ &= \quad -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi + 2\Omega w\cos\phi + \frac{1}{\rho}(\nabla\cdot\tau)_x. \end{aligned} \tag{7.1}$$

For the purposes of Jeffreys' argument, (7.1) can be substantially simplified. Steadiness and zonal symmetry (no eddies) imply $\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = 0$. Scaling shows that

$$\frac{uw}{a}, 2\Omega w\cos\phi \ll \frac{uv\tan\phi}{a}, 2\Omega v\sin\phi$$

(What about the equator?). In addition,

$$v\frac{\partial u}{\partial y} - \frac{uv\tan\phi}{a} = \frac{v}{a}\frac{\partial u}{\partial\phi} - \frac{uv\tan\phi}{a} = \frac{v}{a\cos\phi}\frac{\partial}{\partial\phi}(u\cos\phi).$$

Thus (7.1) becomes

$$\frac{v}{a\cos\phi}\frac{\partial}{\partial\phi}(u\cos\phi) + w\frac{\partial u}{\partial z} - 2\Omega v\sin\phi = \frac{1}{\rho}(\nabla\cdot\tau)_x. \tag{7.2}$$

Jeffreys further set

$$(\nabla\cdot\tau)_x = \frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z}. \tag{7.3}$$

Integrating (7.2) over all heights yields

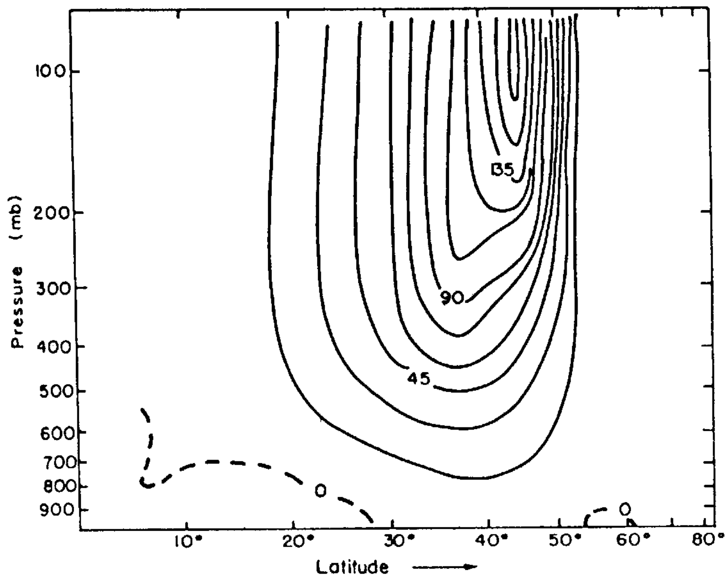
$$\int_0^\infty \frac{\rho v}{a\cos\phi}\frac{\partial}{\partial\phi}(u\cos\phi)\,dz + \int_0^\infty \rho w\frac{\partial u}{\partial z}\,dz$$

$$-2\Omega \sin\phi \int_0^\infty \rho_0 v\, dz = -\mu \left.\frac{\partial u}{\partial z}\right|_0 = -C_D\, u_0^2, \tag{7.4}$$

where C_D is the surface drag coefficient, and $C_D\, u_0^2$ is the usual phenomenological expression for the surface drag. Note that the earth's angular momentum cannot supply momentum removed by surface drag (since there is no net meridional mass flow; i.e., $\int_0^\infty \rho_0 v\, dz = 0.$). Thus, $C_D\, u_0^2$ must be balanced by the advection of relative momentum. Jeffreys argued that the integrals on the left-hand side of (7.4) would be dominated by the first integral evaluated within the first kilometer or so of the atmosphere. Underlying his argument was the obvious lack of data to do otherwise. He then showed that this integral was about a factor of 20 smaller than $C_D\, u_0^2$. He concluded that the maintenance of the surface westerlies had to be achieved by the neglected eddies. It may seem odd that Jeffreys, who so carefully considered the effect of the return v-flow on the Coriolis torque, ignored it for the transport of relative momentum. However, since it was the Ferrel cell he was thinking of, its inclusion would not have altered his conclusion. What he failed to note was that in both Hadley's model and that of Ferrel and Thomson, it was the Hadley cell which supplied westerly momentum to middle latitudes. Thus Jeffreys' argument is totally inconclusive; it certainly is not a proof that a symmetric circulation would be impossible (though this was sometimes claimed in the literature).

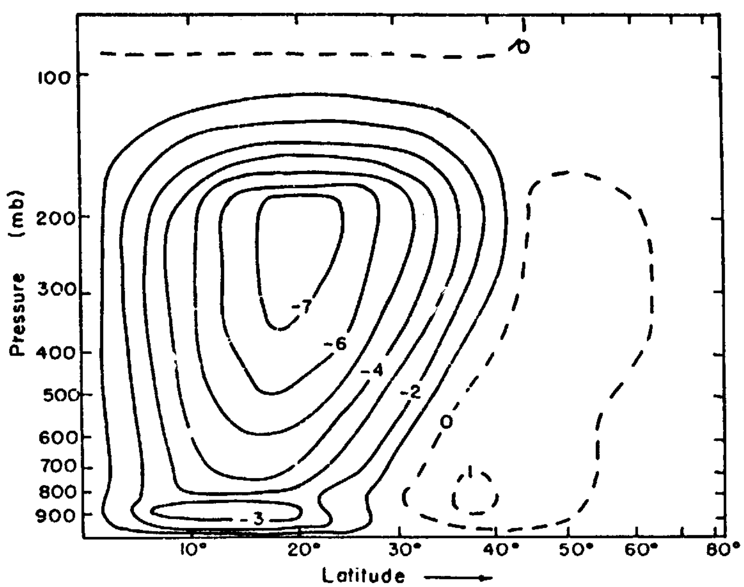
A more balanced view was presented by Villem Bjerknes (1937) towards the end of his career. Bjerknes suggested that in the absence of eddies the atmosphere would have a Ferrel-Thomson circulation – but that such an atmosphere would prove unstable to eddies. This suggestion did not, however, offer any estimate of the extent to which the symmetric circulation could explain the general circulation, and the extent to which eddies are essential.

Ed Schneider and I attempted to answer this question by means of a rather cumbersome numerical calculation (Schneider and Lindzen, 1977; Schneider, 1977). The results shown in Figure 7.3 largely confirm Bjerknes' suggestion. The main shortcoming of this calculation was that it yielded a zonal jet that was much too strong. Surface winds were also a little weaker than observed, but on the whole, the symmetric circulation suffered from none of the inabilities Jeffreys had attributed to it. In order to see how the symmetric circulation works, it is fortunate that Schneider (1977) discovered a rather simple approximate approach to calculating the Hadley circulation. Held and Hou (1980) explored this approximation in some detail. We shall briefly go over the Held and Hou calculations.



100
200
300
400
500
600
700
800
900
Pressure (mb)
135
90
45
0
0
10°
20°
30°
40°
50°
60°
80°
Latitude

(a)



100
200
300
400
500
600
700
800
900
Pressure (mb)
0
-7
-6
-4
-2
0
-3
10°
20°
30°
40°
50°
60°
80°
Latitude

(b)

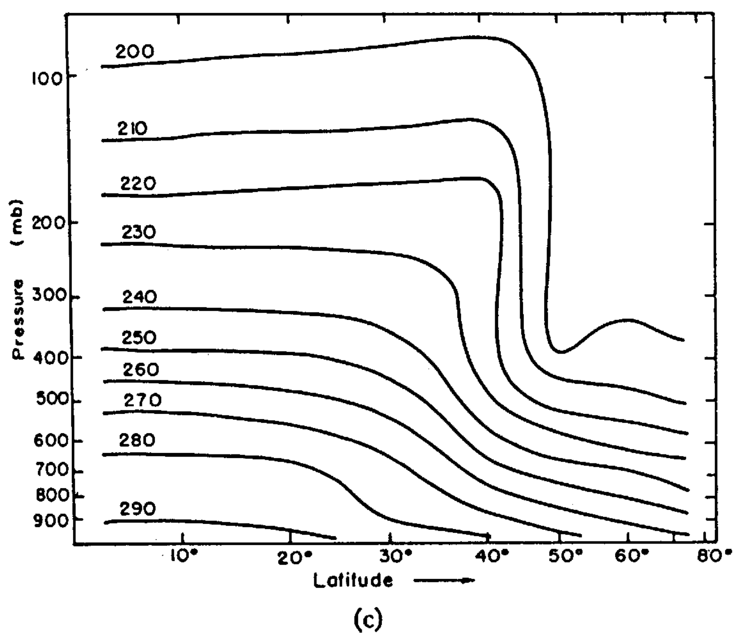


Figure 7.3: Example of an eddy-free symmetric circulation found by Schneider (1977). Panel (a) shows contours of zonal wind (contour intervals of 15ms^{-1}). Panel (b) shows streamfunction contours (contour intervals of 10^{12} gs^{-1}). Panel (c) shows temperature contours (contour intervals of 10 K).

7.2 Held and Hou calculations

Held and Hou restrict themselves to a Boussinesq fluid of depth H. For such a fluid, the continuity equation is simplified to

$$\nabla \cdot \vec{u} = 0. \tag{7.5}$$

With (7.5) as well as the assumptions of steadiness and zonal symmetry, and the retention of only vertical diffusion in the viscous stress and thermal conduction terms, our remaining equations of motion become

$$\nabla \cdot (\vec{u}u) - \underbrace{f}_{2\Omega \sin\phi} v - \frac{uv\tan\phi}{a} = \frac{\partial}{\partial z}\left(\nu \frac{\partial u}{\partial z}\right) \tag{7.6}$$

$$\nabla \cdot (\vec{u}v) + fu + \frac{u^2\tan\phi}{a} = -\frac{1}{a}\frac{\partial \Phi}{\partial \phi} + \frac{\partial}{\partial z}\left(\nu \frac{\partial v}{\partial z}\right) \tag{7.7}$$

$$\nabla \cdot (\vec{u}\Theta) = \frac{\partial}{\partial z}\left(\nu \frac{\partial \Theta}{\partial z}\right) - \frac{(\Theta - \Theta_E)}{\tau} \tag{7.8}$$

and

$$\frac{\partial \Phi}{\partial z} = g\frac{\Theta}{\Theta_0}. \tag{7.9}$$

(N.B. $\Phi = \frac{p}{\rho}$.) The quantity Θ_E is presumed to be a 'radiative' equilibrium temperature distribution for which we adopt the simplified form

$$\frac{\Theta_E(\phi, z)}{\Theta_0} \equiv 1 - \Delta_H\left(\frac{1}{3} + \frac{2}{3}P_2(\sin\phi)\right) + \Delta_V\left(z - \frac{H}{2}\right), \tag{7.10}$$

where $P_2(x) \equiv \frac{1}{2}(3x^2 - 1)$, $x = \sin\phi$, $\Theta_0 = \Theta_E(0, \frac{H}{2})$, Δ_H = fractional potential temperature drop from the equator to the pole, Δ_V = fractional potential temperature drop from H to the ground, and τ is a 'radiative' relaxation time. (The reader should work out the derivation of Equations 7.6–7.9. N.B. In the Boussinesq approximation, density is taken as constant except where it is multiplied by g. The resulting simplifications can also be obtained for a fully stratified atmosphere by using the log-pressure coordinates described in Chapter 4.)

The boundary conditions employed by Held and Hou are

$$\underbrace{\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}}_{no\ stress} = \overbrace{\frac{\partial \Theta}{\partial z}}^{no\ heat\ conduction} = \underbrace{w}_{rigid\ top} = 0 \qquad \text{at } z = H \tag{7.11}$$

$$\frac{\partial \Theta}{\partial z} = w = 0 \qquad \text{at } z = 0 \tag{7.12}$$

$$\underbrace{\nu\frac{\partial u}{\partial z} = Cu, \quad \nu\frac{\partial v}{\partial z} = Cv}_{linearization\ of\ surface\ stress\ conditions} \qquad \text{at } z = 0 \tag{7.13}$$

$$\underbrace{v = 0 \qquad \text{at } \phi = 0}_{symmetry\ about\ the\ equator}\ . \tag{7.14}$$

The quantity C is taken to be a constant drag coefficient. Note that this is not the same drag coefficient that appeared in (7.4); neither is the expression for surface drag which appears in (7.13) the same. As noted in (7.13), the expression is a linearization of the full expression. The idea is that the full expression is quadratic in the *total* surface velocity – of which the contribution of the Hadley circulation is only a part. The coefficient C results from the product of C_D and the 'ambient' surface wind.

When $\nu \equiv 0$ we have already noted that our equations have an exact solution:

$$v = w = 0 \tag{7.15}$$

$$\Theta = \Theta_E \tag{7.16}$$

and

$$u = u_E, \tag{7.17}$$

where u_E satisfies

$$\frac{\partial}{\partial z}\left(fu_E + \frac{u_E^2 \tan\phi}{a}\right) = -\frac{g}{a\Theta_0}\frac{\partial \Theta_E}{\partial \phi} \qquad \text{(where } y = a\phi\text{)}. \tag{7.18}$$

If we set $u_E = 0$ at $z = 0$, the appropriate integral of (7.18) is (see Holton, 1992, p. 67):

$$\frac{u_E}{\Omega a} = \left[\left(1 + 2R\frac{z}{H}\right)^{1/2} - 1\right]\cos\phi, \tag{7.19}$$

where

$$R = \frac{gH\Delta_H}{(\Omega a)^2}. \tag{7.20}$$

When $R \ll 1$,

$$\frac{u_E}{\Omega a} = R\cos\phi\frac{z}{H}. \tag{7.21}$$

Why isn't the above solution, at least, approximately appropriate? Why do we need a meridional solution at all? Hadley already implicitly recognized that the answer lies in the presence of viscosity. A theorem (referred to as 'Hide's theorem') shows that if we have viscosity (no matter how small), (7.15)–(7.19) cannot be a steady solution of the *symmetric* equations.

7.2.1 Hide's theorem and its application

The proof of the theorem is quite simple. We can write the total angular momentum per unit mass as

$$M \equiv \Omega a^2 \cos^2\phi + ua\cos\phi \tag{7.22}$$

(recall that ρ is taken to be 'constant'), and (7.6) may be rewritten

$$\nabla \cdot (\vec{u}M) = \nu\frac{\partial^2 M}{\partial z^2}. \tag{7.23}$$

Now suppose that M has a local maximum somewhere in the fluid. We may then find a closed contour surrounding this point where M is constant. If we integrate (7.23) about this contour, the contribution of the left-hand side will go to zero (Why?), while the contribution of the right-hand side will be negative (due to down gradient viscous fluxes). Since such a situation is inconsistent, M cannot have a maximum in the interior of the fluid. We next consider the possibility that M has a maximum at the surface. We may now draw a constant M contour above the surface, and close the contour along the surface (where $w = u_n = 0$). Again, the contribution from the left-hand side will be zero. The contribution from the right-hand side will depend on the sign of the surface wind. If the surface wind is westerly, then the contribution of the right-hand side will again be negative, and M, therefore, cannot have a maximum at the surface where there are surface westerlies. If the surface winds are easterly, then there is, indeed, a possibility that the contribution from the right-hand side will be zero. Thus, the maximum value of M must occur at the surface in a region of surface easterlies[1]! An upper bound for M is given by its value at the equator when $u = 0$; that is,

$$M_{max} < \Omega a^2. \tag{7.24}$$

Now u_E as given by (7.19) implies (among other things) westerlies at the equator and increasing M with height at the equator – all of which is forbidden by *Hide's theorem* – at least for symmetric circulations. *A meridional circulation is needed in order to produce adherence to Hide's theorem.*

Before proceeding to a description of this circulation we should recall that in our discussion of observations we did indeed find zonally averaged westerlies above the equator (in connection with the quasi-biennial oscillation, for example). This implies the existence of eddies which are transporting angular momentum up the gradient of mean angular momentum!

A clue to how much of a Hadley circulation is needed can be obtained by seeing where $u_E = u_M$; by u_M we mean the value of u associated with $M = \Omega a^2$ (*viz* Equation 7.24). From (7.22) we get

$$u_M = \frac{\Omega a \sin^2 \phi}{\cos \phi}. \tag{7.25}$$

[1]It is left to the reader to show that M cannot have a maximum at a stress-free upper surface.

Setting $u_E = u_M$ gives an equation for $\phi = \phi^*$. For $\phi < \phi^*$, u_E violates Hide's theorem.

$$[(1+2R)^{1/2} - 1]\cos\phi^* = \frac{\sin^2\phi^*}{\cos\phi^*}. \tag{7.26}$$

(Recall that $R \equiv \frac{gH\Delta_H}{(\Omega a)^2}$.)

Solving (7.26) we get

$$\phi^* = \tan^{-1}\{[(1+2R)^{1/2} - 1]^{1/2}\}. \tag{7.27}$$

For small R,

$$\phi^* = R^{1/2}. \tag{7.28}$$

Using reasonable atmospheric values

$$g = 9.8 \text{ ms}^{-2}$$
$$H = 1.5\ 10^4 \text{ m}$$
$$\Omega = 2\pi/(8.64\ 10^4 \text{ s})$$
$$a = 6.4\ 10^6 \text{ m}$$

and

$$\Delta_H \sim 1/3,$$

we obtain from (7.20)

$$R \approx .226$$

and

$$R^{1/2} \approx .48$$

and from (7.28)

$$\phi^* \approx 30^\circ.$$

Thus, we expect a Hadley cell over at least half the globe[2].

7.2.2 Simplified calculations

Solving for the Hadley circulation is not simple even for the highly simplified model of Held and Hou. However, Schneider and Held and Hou discovered that the solutions they ended up with when viscosity was low were approximately constrained by a few principles which served to determine the main features of the Hadley circulation:

1. The upper poleward branch conserves angular momentum;

2. The zonal flow is balanced; and

3. Surface winds are small compared to upper level winds.

   In addition:

4. Thermal diffusion is not of dominant importance in Equation 7.8.

Held and Hou examine, in detail, the degree to which these principles are valid, and you are urged to read their work. However, here we shall merely examine the implications of items (1)–(4) and see how these compare with the numerical solutions from Held and Hou.
Principle (1) implies

$$u(H, \phi) = u_M = \frac{\Omega a \sin^2 \phi}{\cos \phi}. \tag{7.29}$$

[2]The choice $\Delta_H = 1/3$ is taken from Held and Hou (1980) and corresponds to Θ_E varying by about 100° between the equator and the poles. This, indeed, is reasonable for radiative equilibrium. However, more realistically, the atmosphere is, at any moment, more nearly in equilibrium with the sea surface (because adjustment times for the sea are much longer than for the atmosphere) and, therefore, a choice of $\Delta_H \approx 1/6$ may be more appropriate. This leads to $R^{1/2} \approx .34$ and $\phi^* \approx 20^\circ$, which is not too different from what was obtained for $\Delta_H = 1/3$. This relative insensitivity of the Hadley cell extent makes it a fairly poor variable for distinguishing between various parameter choices.

Principle (2) implies

$$fu + \frac{u^2 \tan\phi}{a} = -\frac{1}{a}\frac{\partial \Phi}{\partial \phi}. \tag{7.30}$$

Evaluating (7.30) at $z = H$ and $z = 0$, and subtracting the results yields

$$\begin{aligned} f[u(H) - u(0)] \quad + \quad & \frac{\tan\phi}{a}[u^2(H) - u^2(0)] \\ & = -\frac{1}{a}\frac{\partial}{\partial \phi}[\Phi(H) - \Phi(0)]. \end{aligned} \tag{7.31}$$

Integrating (7.9) from $z = 0$ to $z = H$ yields

$$\frac{\Phi(H) - \Phi(0)}{H} = \frac{g}{\Theta_0}\bar{\Theta}, \tag{7.32}$$

where $\bar{\Theta}$ is the vertically averaged potential temperature. Substituting (7.32) into (7.31) yields a simplified 'thermal wind' relation

$$f[u(H) - u(0)] + \frac{\tan\phi}{a}[u^2(H) - u^2(0)] = -\frac{gH}{a\Theta_0}\frac{\partial \bar{\Theta}}{\partial \phi}. \tag{7.33}$$

Principle (3) allows us to set $u(0) = 0$. Using this and (7.29), (7.33) becomes

$$2\Omega \sin\phi \frac{\Omega a \sin^2\phi}{\cos\phi} + \frac{\tan\phi}{a}\frac{\Omega^2 a^2 \sin^4\phi}{\cos^2\phi} = -\frac{gH}{a\Theta_0}\frac{\partial \bar{\Theta}}{\partial \phi}. \tag{7.34}$$

Equation 7.34 can be integrated with respect to ϕ to obtain

$$\frac{\bar{\Theta}(0) - \bar{\Theta}(\phi)}{\Theta_0} = \frac{\Omega^2 a^2}{gH}\frac{\sin^4\phi}{2\cos^2\phi}. \tag{7.35}$$

Note that conservation of angular momentum and the maintenance of a balanced zonal wind completely determine the variation of $\bar{\Theta}$ within the Hadley regime. Moreover, the decrease of $\bar{\Theta}$ with latitude is much slower near the equator than would be implied by Θ_E!

Finally, we can determine both $\bar{\Theta}(0)$ and the extent of the Hadley cell, ϕ_H, with the following considerations:

1. At ϕ_H, temperature should be continuous so

$$\bar{\Theta}(\phi_H) = \bar{\Theta}_E(\phi_H). \tag{7.36}$$

2. From Equation 7.8 we see that the Hadley circulation does not produce net heating over the extent of the cell. For the diabatic heating law in Equation 7.8 we therefore have

$$\int_0^{\phi_H} \bar{\Theta} \cos\phi \, d\phi = \int_0^{\phi_H} \bar{\Theta}_E \cos\phi \, d\phi. \tag{7.37}$$

Substituting (7.35) into (7.36) and (7.37) yields the two equations we need in order to solve for ϕ_H and $\bar{\Theta}(0)$. The solution is equivalent to matching (7.35) to $\bar{\Theta}_E$ so that 'equal areas' of heating and cooling are produced. This is schematically illustrated in Figure 7.4. Also shown are $u_M(\phi)$ and $u_E(\phi)$.

The algebra is greatly simplified by assuming small ϕ. Then (7.35) becomes

$$\frac{\bar{\Theta}}{\Theta_0} \approx \frac{\bar{\Theta}(0)}{\Theta_0} - \frac{1}{2}\frac{\Omega^2 a^2}{gH}\phi^4 \tag{7.38}$$

and (7.10) becomes

$$\frac{\bar{\Theta}_E}{\Theta_0} = \frac{\bar{\Theta}_E(0)}{\Theta_0} - \Delta_H \phi^2. \tag{7.39}$$

Substituting (7.38) and (7.39) into (7.36) and (7.37) yields

$$\frac{\Theta(0)}{\Theta_0} = \frac{\bar{\Theta}_E(0)}{\Theta_0} - \frac{5}{18} R\Delta_H \tag{7.40}$$

and

$$\phi_H = \left(\frac{5}{3}R\right)^{1/2}. \tag{7.41}$$

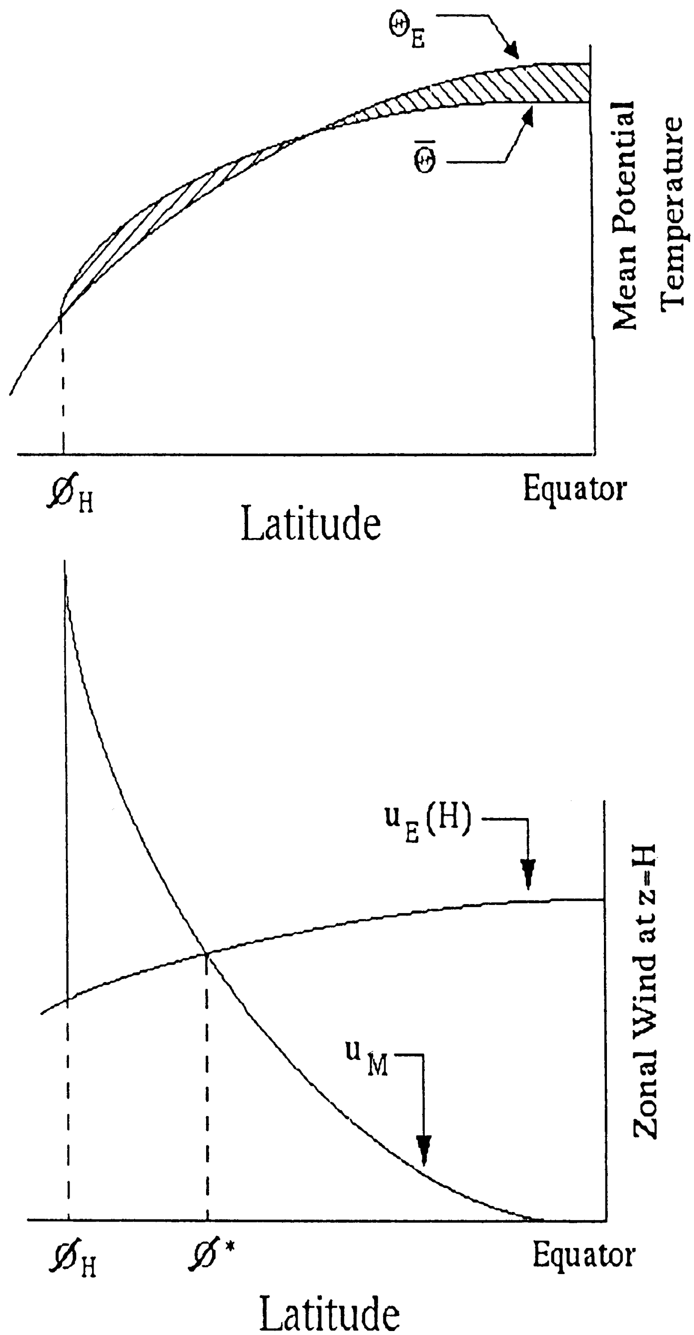


Figure 7.4: Schematic drawings of the vertical mean potential temperature distribution (upper figure) and the zonal wind distribution at the top of the atmosphere (lower figure). With Newtonian cooling (linear in Θ), conservation of potential temperature requires that the shaded areas be equal. Note that this idealized circulation increases the baroclinicity of the flow between ϕ^* (where $u_E = U_M$) and ϕ_H.

(Remember that $R \equiv \frac{gH\Delta_H}{\Omega^2 a^2}$; for a slowly rotating planet such as Venus, ϕ_H can extend to the pole.)

We see that continuity of potential temperature and conservation of angular momentum and potential temperature serve to determine the meridional distribution of temperature. the intensity of the Hadley circulation will be such as to produce this temperature distribution. In section 10.2 of Houghton, 1977, there is a description of Charney's viscosity dominated model for a meridional circulation. In that model, $\Theta = \Theta_E$ and $u = u_E$, except in thin boundary layers, and the meridional velocity is determined by requiring that fv balance the viscous diffusion of momentum, $\nu \frac{\partial^2 u}{\partial z^2}$. Such a model clearly violates Hide's theorem. A more realistic viscous model is described by Schneider and Lindzen (1977) and Held and Hou (1980), wherein the meridional circulation is allowed to modify Θ through the following linearization of the thermodynamic energy equation

$$-w\frac{\partial \Theta}{\partial z} = (\Theta - \Theta_E)/\tau,$$

where $\frac{\partial \Theta}{\partial z}$ is a specified constant. In such models the meridional circulation continues, with gradual diminution, to high latitudes rather than ending abruptly at some subtropical latitude – as happens in the present 'almost inviscid' model. On the other hand, the modification of Θ (for the linear viscous model) is restricted to a neighborhood of the equator given by

$$\phi \sim R^{*1/4},$$

where

$$R^* \equiv \left(\frac{\tau\nu}{4H}\right)\left(\frac{gH}{\Omega^2 a^2}\right)\Delta_V.$$

When $R^{*1/4} \geq R^{1/2}$ (*viz.* Equation 7.28) then a viscous solution can be compatible with Hide's theorem. Note, however, that linear models cannot have surface winds (Why? Hint: consider the discussion of Jeffreys' argument.).

Returning to our present model, there is still more information which can be extracted. Obtaining the vertically integrated flux of potential temperature is straightforward.

$$\frac{1}{H}\int_0^H \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(v\Theta\cos\phi)\,dz = \frac{\bar{\Theta}_E - \bar{\Theta}}{\tau}. \tag{7.42}$$

In the small ϕ limit, $\bar{\Theta}_E$ and $\bar{\Theta}$ are given by (7.38)–(7.41) and (7.42) can be integrated to give

$$\frac{1}{\Theta_0}\int_0^H v\Theta\,dz = \frac{5}{18}\left(\frac{5}{3}\right)^{1/2}\frac{Ha\Delta_H}{\tau}R^{3/2}\left[\frac{\phi}{\phi_H} - 2\left(\frac{\phi}{\phi_H}\right)^3 + \left(\frac{\phi}{\phi_H}\right)^5\right]. \tag{7.43}$$

Held and Hou are also able to estimate surface winds on the basis of this simple model. For this purpose additional assumptions are needed:

1. One must assume either
   (a) the meridional flow is primarily confined to thin boundary layers adjacent to the two horizontal boundaries, or that
   (b) profiles of u and Θ are self-similar so that

   $$\frac{u(z)-u(0)}{u(H)-u(0)} \approx \frac{\Theta(z)-\Theta(0)}{\Theta(H)-\Theta(0)}.$$

   (We shall employ (a) because it's simpler.)

2. Neither the meridional circulation nor diffusion affects the static stability so that

   $$\frac{\Theta(H)-\Theta(0)}{\Theta_0} \approx \Delta_V.$$

   (This requires that the circulation time and the diffusion time both be longer than τ; a serious discussion of this would require consideration of cumulus convection.)

With assumptions (1) and (2) above we can write

$$\frac{1}{\Theta_0}\int_0^H v\Theta\,dz \approx V\Delta_V, \tag{7.44}$$

where V is a mass flux in the boundary layers. With (7.44), (7.43) allows us to solve for V.

Similarly, we have for the momentum flux

$$\int_0^H vu\,dz \approx Vu_M. \tag{7.45}$$

To obtain the surface wind, we vertically integrate (7.23) (using (7.22) and (7.13)) to get

$$\frac{1}{a\cos^2\phi}\frac{\partial}{\partial\phi}\left(\cos^2\phi\int_0^H uv\,dz\right) = -Cu(0). \tag{7.46}$$

From (7.43)–(7.46) we then get

$$Cu(0) \approx -\frac{25}{18}\frac{\Omega aH\Delta_H}{\tau\Delta_V}R^2\left[\left(\frac{\phi}{\phi_H}\right)^2 - \frac{10}{3}\left(\frac{\phi}{\phi_H}\right)^4 + \frac{7}{3}\left(\frac{\phi}{\phi_H}\right)^6\right]. \tag{7.47}$$

Equation 7.47 predicts surface easterlies for

$$\phi < \left(\frac{3}{7}\right)^{1/2}\phi_H \tag{7.48}$$

and westerlies for

$$\left(\frac{3}{7}\right)^{1/2}\phi_H < \phi < \phi_H. \tag{7.49}$$

For the parameters given following Equation 7.28, the positions of the upper level jet and the easterlies and westerlies are moderately close to those observed. It can also be shown that for small ν, the above scheme leads to a Ferrel cell above the surface westerlies. We will return to this later. For the moment we wish to compare the results of the present simple analysis with the results of numerical integrations of Equations 7.5–7.14.

7.2.3 Comparison of simple and numerical results

Unfortunately, the results in Held and Hou are for $H = 8 \times 10^3$m rather than 1.5×10^4m. From our simple relations, we correctly expect this to cause features to be compressed towards the equator. Also, Held and Hou adopted the following values for τ, C, and Δ_V:

$$\begin{aligned} \tau &= 20 \text{ days} \\ C &= 0.005 \text{ ms}^{-1} \\ \Delta_V &= 1/8. \end{aligned}$$

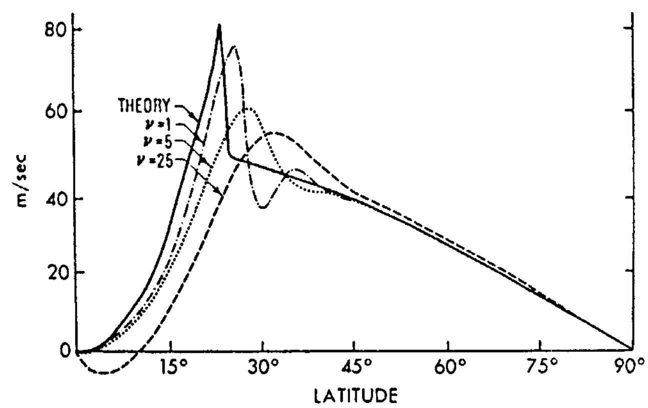


Figure 7.5: Zonal wind at $z = H$ for three values of ν, compared with the simple model for the limit $\nu \to 0$ (from Held and Hou, 1980

Figures 7.5–7.8 compare zonal winds at $z = H$, M at $z = H$, heat flux, and surface winds for our simple calculations and for numerical integrations with various choices of ν. In general, we should note the following:

1. As we decrease ν the numerical results more or less approach the simple results, and for $\nu = .5\,\text{m}^2\text{s}^{-1}$ (generally accepted as a 'small' value) our simple results are a decent approximation. (In fact, however, reducing ν much more does not convincingly show that the limit actually is reached since the numerical solutions become unsteady.)

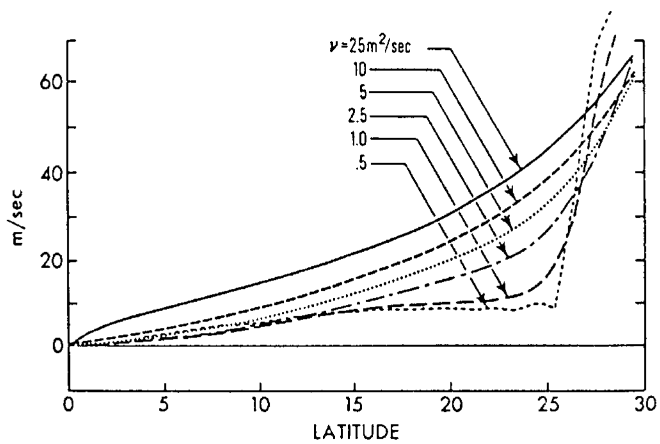


Figure 7.6: A measure of M, namely, $((\Omega a^2 - M)/a)$, evaluated at $z = H$ as a function of ϕ for diminishing values of viscosity, ν. Note that zero corresponds to conservation of M (from Held and Hou, 1980).

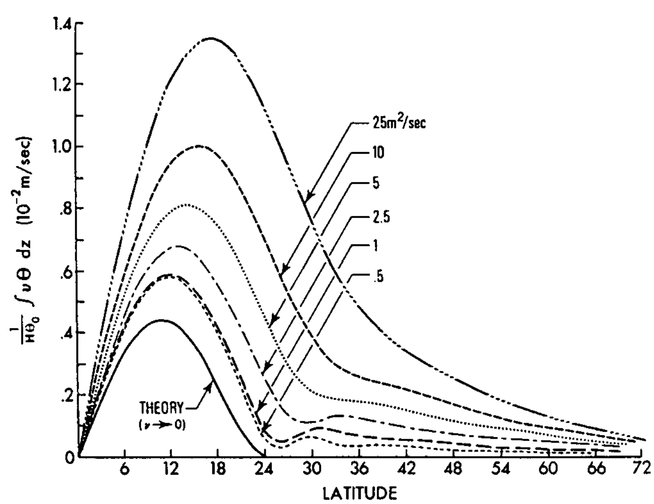


Figure 7.7: Meridional heat fluxes for various values of ν – as well as the theoretical limit based on the simple calculations (from Held and Hou, 1980).

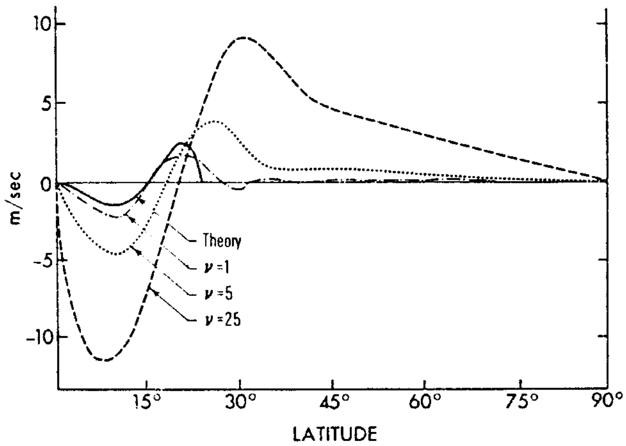


Figure 7.8: Surface wind for various values of ν, and the theoretical inviscid limit based on simple calculations (from Held and Hou, 1980).

2. The presence of modest vertical viscosity increass and broadens both heat flux and the distribution of surface winds (Why?). Viscosity also reduces the magnitude of, broadens, and moves poleward the upper level jet.

3. Very near the equator, the numerical results do not quite converge to constant M at $z = H$. The reason for this can be seen in Figures 7.9 and 7.10 where meridional cross sections are shown for the meridional stream functions and zonal wind.

   Note that the upward branch of the Hadley cell does not rise solely at the equator (as supposed in the simple theory) but over a $10 - -15°$ neighbourhood of the equator. Note also the emergence of the Ferrel cell at small ν.

7.3 Summary and difficulties

Before summarizing what all this tells us about the general circulation let us return to Figure 7.3. We see that Schneider's symmetric circulation, which is by and large consistent with Held and Hou's, also manages to predict an elevated tropical tropopause height and the associated tropopause 'break' at

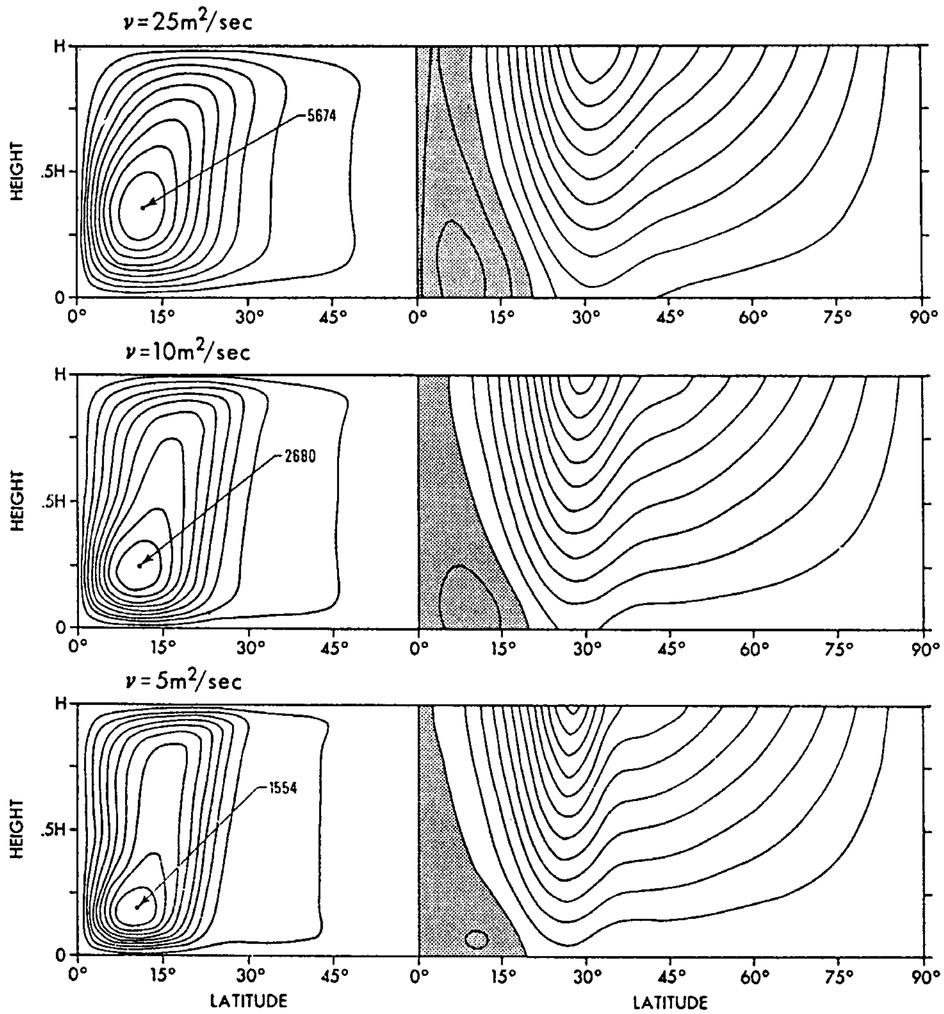


Figure 7.9: Meridional streamfunctions and zonal winds. In the left part of the figure, the streamfunction ψ is given for $\nu = 25, 10,$ *and* $5\,\mathrm{m^2s^{-1}}$, with a contour interval of $0.1\ \psi_{max}$. The value of $0.1\ \psi_{max}\ (\mathrm{m^2s^{-1}})$ is marked by a pointer. The right part of each panel is the corresponding zonal wind field with contour intervals of $5\,\mathrm{ms^{-1}}$. The shaded area indicates the region of easterlies (from Held and Hou, 1980).

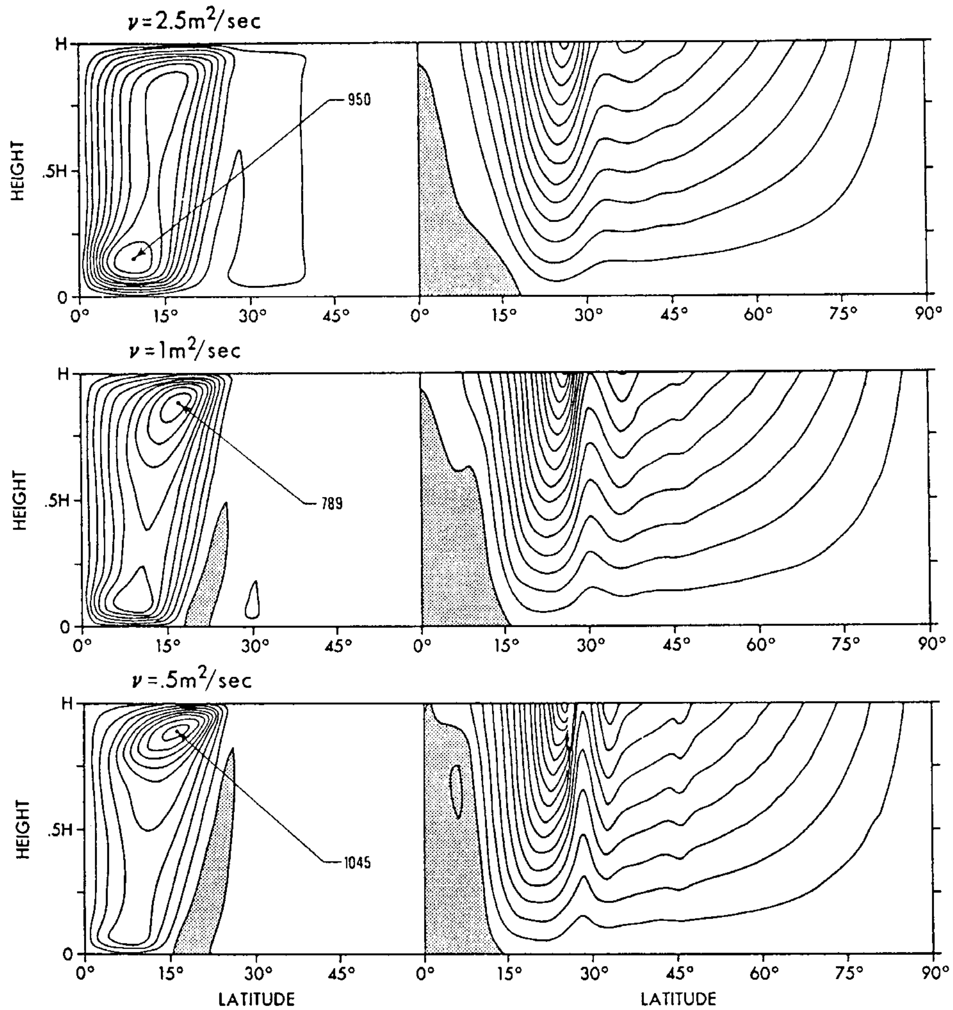


Figure 7.10: Same as Figure 7.9, but for $\nu = 2.5,\ 1.0,\ and\, 0.5\,\mathrm{m^2s^{-1}}$. Note the emergence of a Ferrel cell in the ψ-field where $\psi < 0$ (indicated by shading) (from Held and Hou, 1980).

the edge of the Hadley circulation. The midlatitude tropopause somewhat artificially reflects the assumed Θ_E distribution. The elevated tropopause in the tropics results from the inclusion of cumulus heating.

7.3.1 Remarks on cumulus convection

Cumulus heating will not be dealt with in these lectures, but for the moment three properties of cumulus convection should be noted:

1. It is, in practice, the primary mechanism for carrying heat from the surface in the tropics.

2. Cumulus towers, for simple thermodynamic reasons, extend as high as 16 km, and appear to be the determinant of the tropical tropopause height and the level of Hadley outflow. Remember that tropical circulations tend to wipe out horizontal gradients. Thus, the tropopause tends to be associated with the height of the deepest clouds.

3. Cumulus convection actively maintains a dry static stability (as required in the calculation of Hadley transport). This is explained in Sarachik (1985).

A more detailed description of cumulus convection (and its parameterization) can be found in Emanuel (1994)as well as in Lindzen (1988b).

7.3.2 Preliminary summary

On the basis of our study of symmetric circulations (so far) we find the following:

1. Symmetric solutions yield an upper level jet in about the right place but with much too *large* a magnitude.

2. Symmetric circulations yield surface winds of the right sign in about the right place. In the absence of vertical diffusion, magnitudes are too small, but modest amounts of vertical diffusion corrects matters, and cumulus clouds might provide this ‘diffusion’. (Schneider and Lindzen, 1976, discuss cumulus friction.)

3. Calculated Hadley circulations have only a finite extent. In contrast to Hadley's and Ferrel and Thomson's diagnostic models, the upper branch does not extend to the poles. Thus our Hadley circulation cannot carry heat between the tropics and the poles, and cannot produce the observed pole–equator temperature difference.

4. The calculated temperature distribution does not have the pronounced equatorial minimum at tropopause levels that is observed (*viz.* Figure 5.11).

5. Although not remarked upon in detail, the intensity of the Hadley circulation shown in Figure 7.10 is weaker than what is observed.

At this point we could glibly undertake to search for the resolution of the above discrepancies in the rôle of the thus far neglected eddies. However, before doing this, it is important to ask whether our symmetric models have not perhaps been inadequate in some other way besides the neglect of eddies.

7.4 Asymmetry about the equator

Although we do not have the time to pursue this (and most other matters) adequately, the reader should be aware that critical reassessments are essential to the scientific enterprise – and frequently the source of truly important problems and results. What is wrong with our results is commonly more important than what is right! A particular shortcoming will be discussed here: namely, the assumption that annual average results can be explained with a model that is symmetric about the equator. The importance of this shortcoming has only recently been recognized. (This section is largely based on the material in a paper by Lindzen and Hou, 1988.) This fact alone should encourage the reader to adopt a more careful and critical attitude.

What is at issue in the symmetry assumption can most easily be seen by looking at some data for the meridional circulation itself. Thus far we have not paid too much attention to this field. Figure 7.11 shows the meridional circulation for solstitial conditions. Not surprisingly, it is not symmetric about the equator. More surprising, however, is the degree of asymmetry: the 'winter' cell extends from well into the summer hemisphere ($\sim 20°$) to well into the winter hemisphere ($\sim 30°$), whereas the 'summer' cell barely

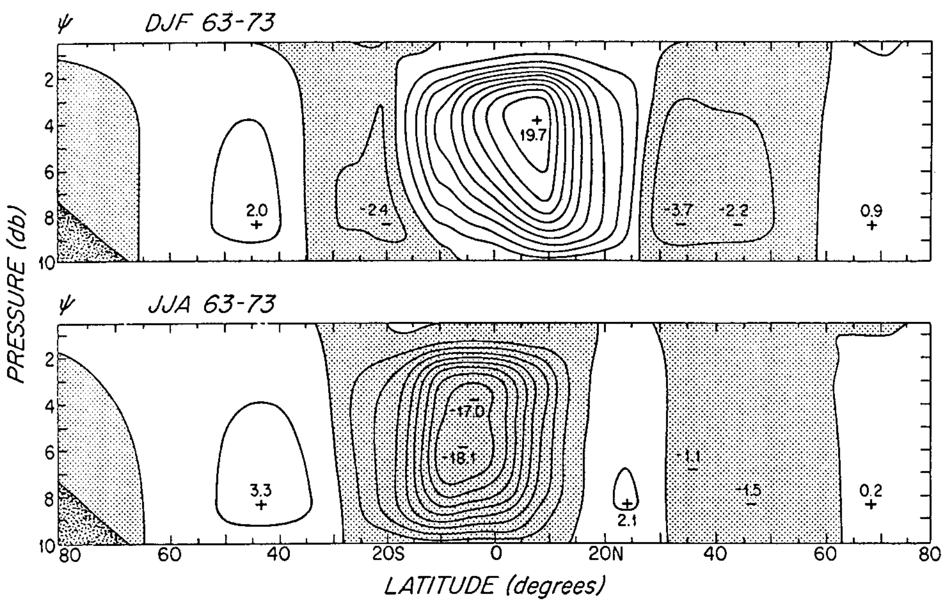


Figure 7.11: Time average meridional-height cross sections of the streamfunction for the mean meridional circulation. Units, 10^{13} gs^{-1}; contour intervals, 0.2×10^{13} gs^{-1}. December–February 1963–73 (upper panel) and June–August 1963–73 (lower panel) (from Oort, 1983).

exists at all! Figures 5.10 and 5.11 show meridional sections of zonally averaged zonal wind and temperature for solstitial conditions. Within about 20–25° of the equator these fields are symmetric about the equator. Thus, in this region, at least, an average over winter and summer of these fields will still give the solstitial distributions. Finally, Figure 7.12 shows the monthly means of the meridional circulation for each of the twelve months of the year. This figure is a little hard to interpret since it extends to only 15°S. However, to a significant extent it suggests that except for the month of April, the asymmetric solstitial pattern is more nearly characteristic of every month than is the idealized symmetric pattern invoked since Hadley in the eighteenth century. Clearly, the assumption of such symmetry is suspect.

Some insight into what is going on can, interestingly enough, be gotten from the simple 'equal area' argument. This is shown in Lindzen and Hou (1988). We will briefly sketch their results here. They studied the axially symmetric response to heating centered off the equator at some latitude ϕ_0. Thus Equation 7.10 was replaced by

$$\frac{\Theta_E}{\Theta_0} \cong 1 + \frac{\Delta_H}{3}(1 - 3(\sin\phi - \sin\phi_0)^2) + \Delta_V \left(\frac{z}{H} - \frac{1}{2}\right). \tag{7.50}$$

The fact that $\phi_0 \neq 0$ substantially complicates the problem. Now the northward and southward extending cells will be different. Although we still require continuity of temperature at the edge of each cell, the northward extent of the Hadley circulation, ϕ_{H+}, will no longer have the same magnitude as the southward extent, $-\phi_{H-}$. Moreover, the 'equal area' argument must now be applied separately to the northern and southern cells. Recall that in the symmetric case, the requirement of continuity at ϕ_H and the requirement of no net heating (i.e., 'equal area') served to determine both ϕ_H and $\bar{\Theta}(0)$, the temperature at the latitude separating the northern and southern cells – which for the symmetric case is the equator. In the present case, this separating latitude can no longer be the equator. If we choose this latitude to be some arbitrary value, ϕ_1, then the application of temperature continuity and 'equal area' for the northern cell will lead to a value of $\bar{\Theta}(\phi_1)$ that will, in general, be different from the value obtained by application of these same constraints to the southern cell. In order to come out with a unique value for ϕ_1 we must allow ϕ_1 to be a variable to be determined.

The solution is now no longer obtainable analytically, and must be determined numerically. This is easily done with any straightforward search

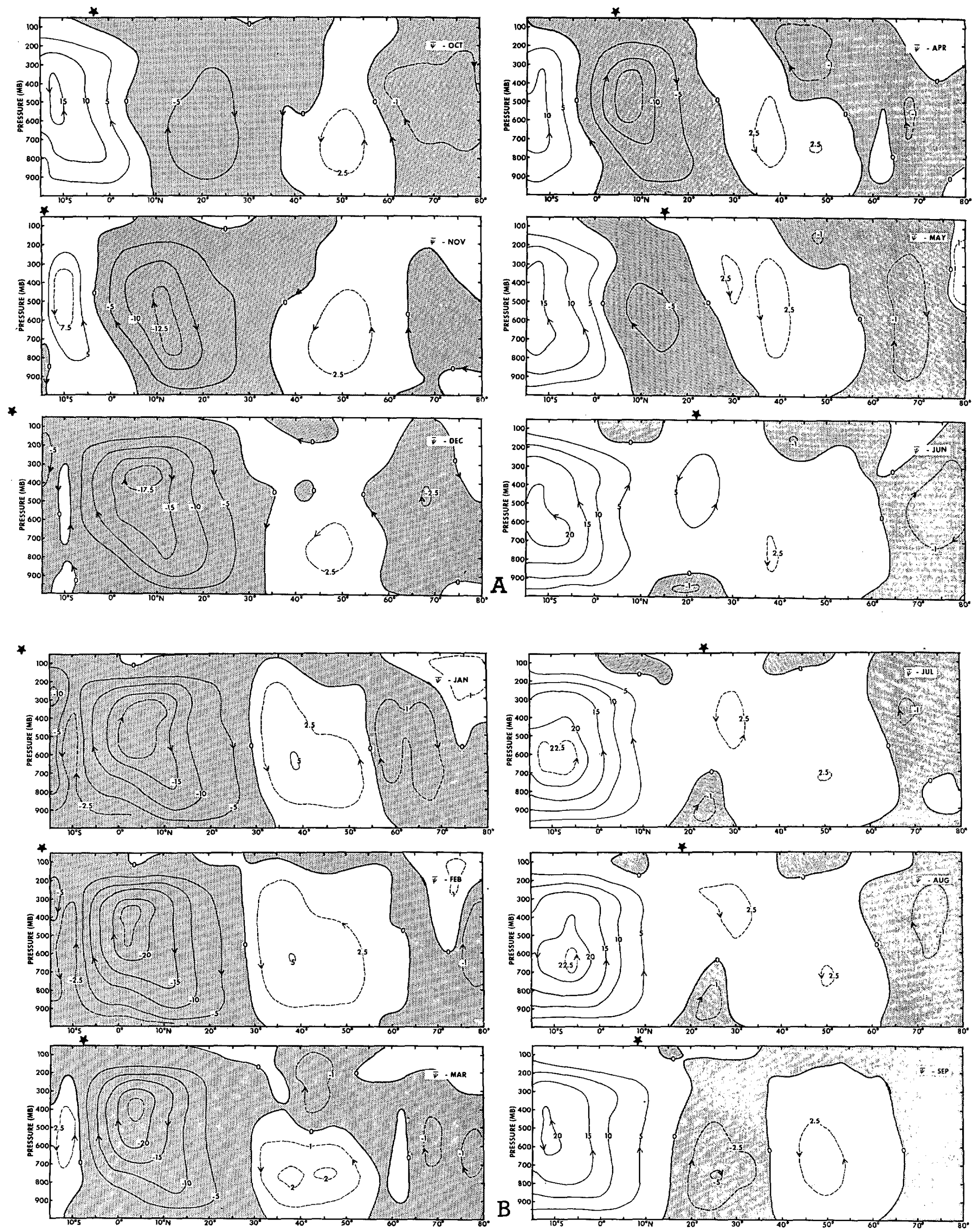


Figure 7.12: Streamlines of the mean meridional circulation for each month. The isolines give the total transport of mass northward below the level considered. Units, $10^{13}\,\mathrm{gs}^{-1}$ (from Oort and Rasmussen, 1970).

routine. Here we will merely present a few of the results. In Figure 7.13 we show how ϕ_{H+}, ϕ_{H-}, and ϕ_1 vary with ϕ_0 for $\Delta_H = 1/3$ (corresponding to a pole–equator temperature difference in $\bar{\Theta}_E$ of about 100°C) and for $\Delta_H = 1/6$. The latter case corresponds to the atmosphere being thermally forced by the surface temperature, and is probably more appropriate for comparisons with observations. For either choice, we see that ϕ_1 goes to fairly large values for small values of ϕ_0. At the same time, ϕ_{H-} also grows to large values while ϕ_{H+} and ϕ_1 asymptotically approach each other – consistent with the northern cell becoming negligible in northern summer. Figure 7.14 shows $\bar{\Theta}$ and $\bar{\Theta}_E$ versus latitude for $\phi_0 = 0$ and $\phi_0 = 6°$. We see very clearly the great enlargement and intensification of the southern cell and the corresponding reduction of the northern cell that accompanies the small northward excursion of ϕ_0 (Recall that the intensity of the Hadley circulation is proportional to $(\overline{\Theta} - \overline{\Theta}_E)$; *viz* Equation 7.42.). We see, moreover, that in agreement with observations at tropopause levels $\bar{\Theta}$ is symmetric about the equator (at least in the neighbourhood of the equator). We also see that $\bar{\Theta}$ has a significant minimum at the equator; such a minimum is observed at the tropopause, but is *not* characteristic of Θ averaged over the depth of the troposphere.

While the simple 'equal area' argument seems to appropriately explain why the Hadley circulation usually consists in primarily a single cell transporting tropical air into the winter hemisphere, the picture it leads to is not without problems. Figure 7.15 shows $u(H, \phi)$ for $\phi_0 = .1$ and $\Delta_H = 1/6$. Consistent with observations, $u(H, \phi_{H+})$ is much weaker than $u(H, \phi_{H-})$ and u is symmetric about the equator in the neighbourhood of the equator, but $u(H, \phi_{H-})$ is still much larger than the observed value, and now $u(H, 0)$ indicates much stronger easterlies than are ever observed. Further difficulties emerge when we look at the surface wind in Figure 7.16. We see that there is now a low level easterly jet on the winter side of the equator; this is, in fact, consistent with observations. However, the surface wind magnitudes (for $\phi_0 = 6°$) are now excessive (only partly due to the linearization of the drag boundary condition), and, more ominously, there are surface westerlies at the equator in violation of Hide's theorem. There are exercises where you are asked to discuss these discrepancies. Lindzen and Hou (1988) show that all these discrepancies disappear in a continuous numerical model with a small amount of viscosity. The discrepancies arise from the one overtly incorrect assumption in the simple approach: namely, that the angular momentum on the upper branch of the Hadley circulation is characteristic of

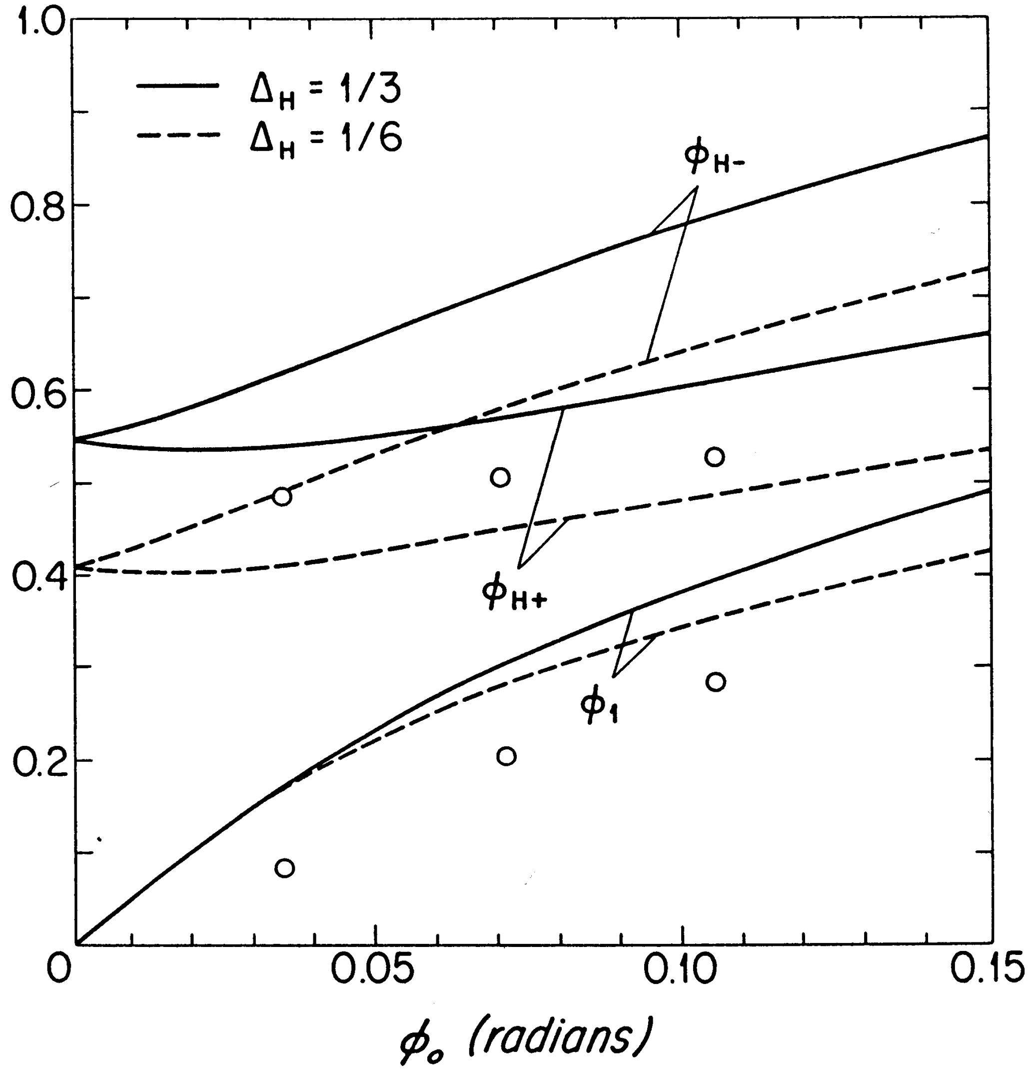


Figure 7.13: The quantities ϕ_{H+}, ϕ_{H-}, and ϕ_1 as functions of ϕ_0 (see text for definitions). Note that 1° of latitude ≅ 0.0175 radians (from Lindzen and Hou, 1988).

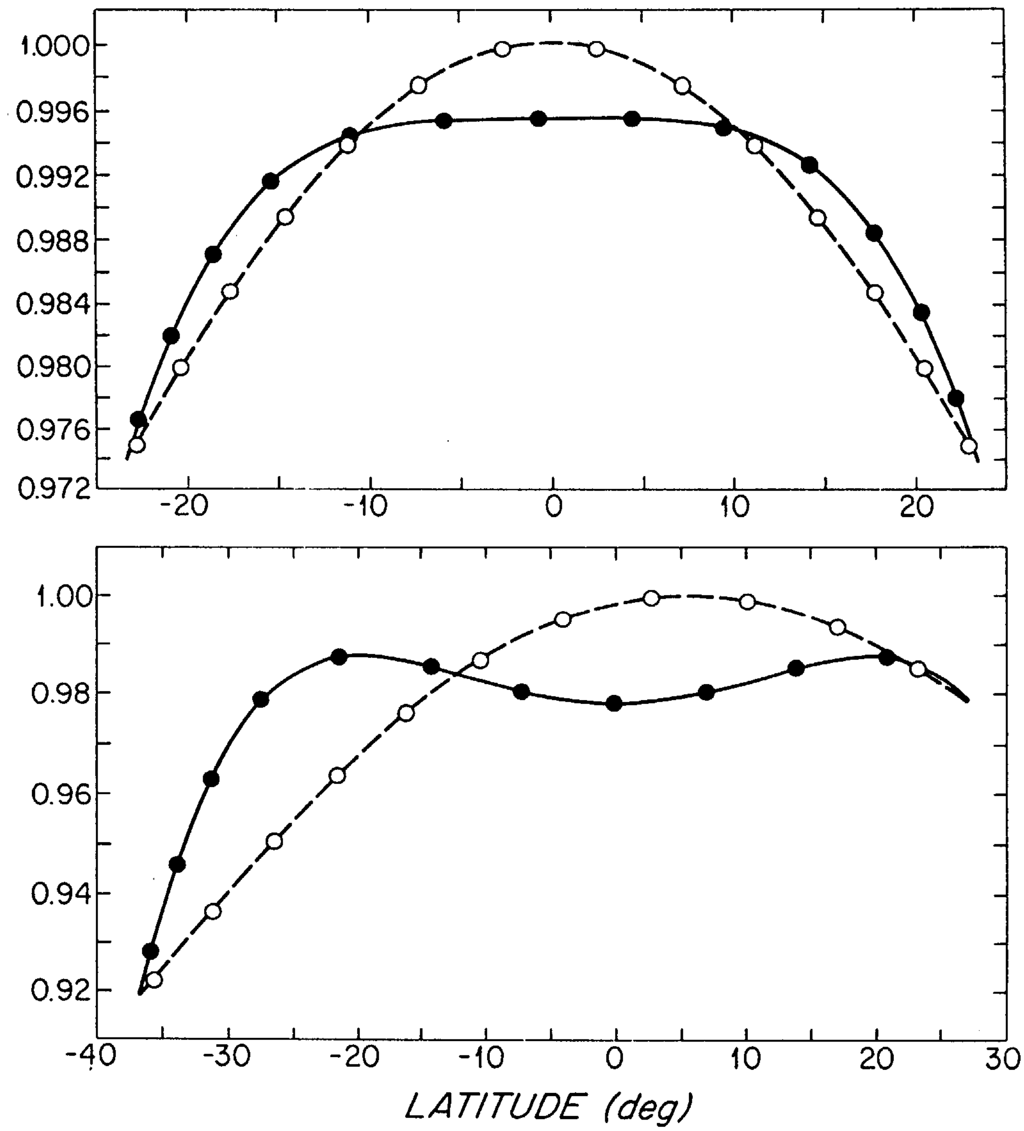


Figure 7.14: $\bar{\Theta}/\Theta_0$ (open circles) and $\bar{\Theta}_E/\Theta_0$ (filled circles) as functions of ϕ obtained with the simple 'equal area' model with $\Delta_H = 1/6$. The upper panel corresponds to $\phi_0 = 0$; the lower panel corresponds to $\phi_0 = 6^o$ (from Lindzen and Hou, 1988).

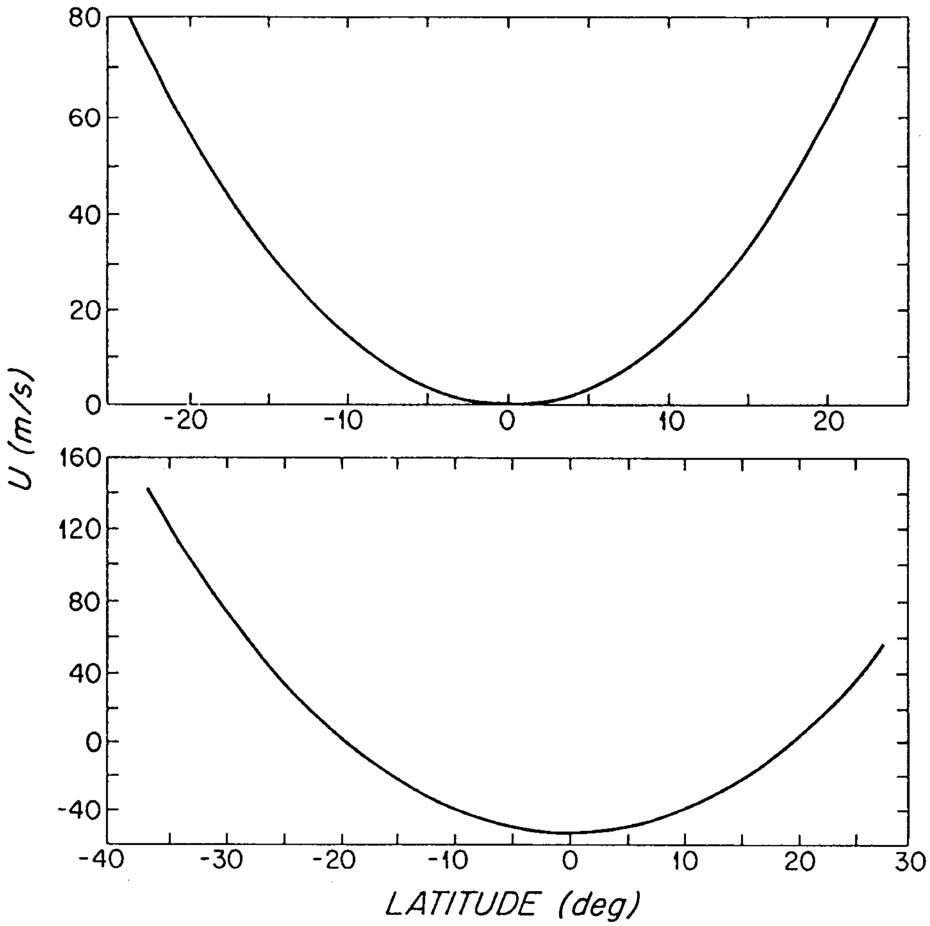


Figure 7.15: Same as Figure 7.14, but for $u(H, \phi)$ (from Lindzen and Hou, 1988).

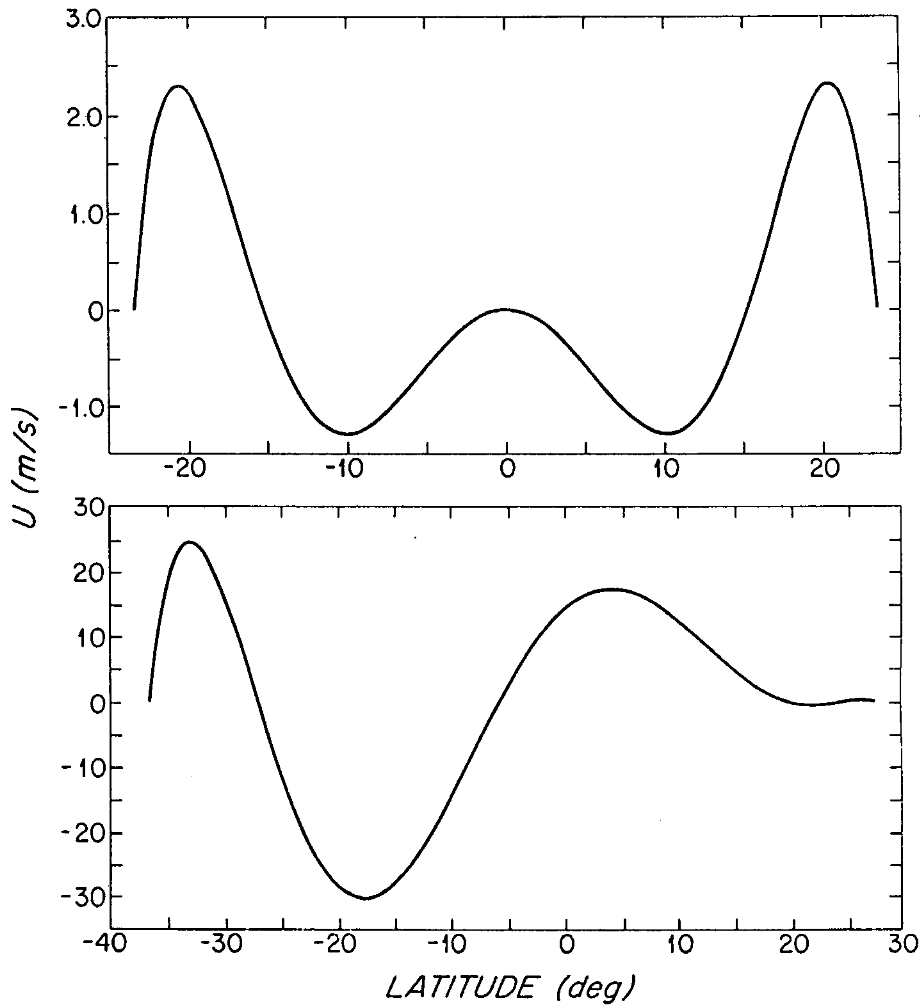


Figure 7.16: Same as Figure 7.14, but for $u(0, \phi)$.

$\phi = \phi_1$ (the latitude separating the northern and southern cells). As we saw in connection with the symmetric Hadley circulation (i.e., $\phi_1 = 0$), the angular momentum in the upper branch was actually characteristic of the entire ascending region. This was not such a significant issue in the symmetric case because the vertical velocity was a maximum at $\phi = \phi_1 = \phi_0 = 0°$. However, when $\phi_0 \neq 0$, then the maximum ascent no longer occurs at ϕ_1; rather it occurs near ϕ_0 where the characteristic angular momentum differs greatly from that at ϕ_1. It should also be mentioned that in the continuous models, the temperature minimum at the equator is substantially diminished.

The above discussion leads to only modest changes in the five points mentioned in Section 7.3.2. Item 5 is largely taken care of when one recognizes that the Hadley circulation resulting from averaging winter and summer circulations is much larger than the circulation produced by equinoctial forcing. Eddies are probably still needed for the following:

1. to diminish the strength of the jet stream, and, relatedly, to maintain surface winds in middle and high latitudes; and

2. to carry heat between the tropics and the poles.

Lindzen and Hou (1988) stress that Hadley circulations mainly transport angular momentum into the winter hemisphere. Thus, to the extent that eddies are due to the instability of the jet, eddy transports are likely to be mostly present in the winter hemisphere.

In this chapter we have seen how studying the symmetric circulation can tell us quite a lot about the real general circulation – even though a pure symmetric circulation is never observed. In the remainder of this volume, we will focus on the nature of the various eddies. Our view is that eddies are internal waves interacting with the 'mean flow'. Forced waves lose energy to the mean flow, while unstable waves gain energy at the expense of the mean flow.