Chapter 6

Equations of motion

Supplemental reading:

Holton (1979), chapters 2 and 3 deal with equations, section 2.3 deals with spherical coordinates, section 2.4 deals with scaling, and section 3.1 deals with pressure coordinates.

Houghton (1977), Chapter 7 deals with equations, and Section 7.1 deals with spherical coordinates.

Serrin (1959)

As has been mentioned in the Introduction, it is expected that almost everyone reading these lecture notes (and despite TeXification, these are only notes) will have already seen a derivation of the equations. I have, therefore, decided to cover the equations using Serrin's somewhat less familiar approach.

6.1 Coordinate systems and conservation

Let $\vec{x} = (x_1, x_2, x_3)$ be a fixed spatial position; this will be referred to as an *Eulerian* coordinate system. Now, at some moment $t = 0$ let's look at a fluid and label each particle of the fluid $\vec{X} = \vec{X}(t, \vec{x}) = (X_1, X_2, X_3)$, where $X_i|_{t=0} = x_i$; that is, we label each particle by its position at $t = 0$; this will

be referred to as a *Lagrangian* coordinate system. In general, each coordinate system may, in principle, be transformed into the other:

$$\text{Eulerian:} \qquad \vec{x}, t \quad \vec{x} = \vec{x}(\vec{X}, t)$$
$$\text{Lagrangian:} \qquad \vec{X}, t \quad \vec{X} = \vec{X}(\vec{x}, t)$$

Let the velocity of a fluid 'particle' be $\vec{u} = (u_1, u_2, u_3)$.

$$u_i = \frac{Dx_i}{Dt} = \left(\frac{\partial x_i}{\partial t}\right)_{\vec{X} \ constant}.$$

Similarly, let $\vec{a}$ be the acceleration of a fluid 'particle':

$$a_i = \left(\frac{\partial^2 x_i}{\partial t^2}\right)_{\vec{X} \ constant} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j}\frac{Dx_j}{Dt},$$

where the *summation convention* is used; that is, we sum over repeated indices.

The laws of physics are fundamentally conservation statements concerning $\frac{D}{Dt}$ of something following the fluid. Let us, for the moment, deal with some unspecified field $f(x_i, t)$ (per unit mass):

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}\frac{Dx_i}{Dt}.$$

Now consider some region of space $R(\vec{x})$. We wish to evaluate

$$\frac{D}{Dt}\int_{R(\vec{x})} f\rho\, d^3x,$$

where ρ is density. A difficulty arises since the fluid within $R(\vec{x})$ is changing. We deal with this by switching to Lagrangian coordinates:

$$\frac{D}{Dt}\int_{R(\vec{x})} f\rho\, d^3x = \frac{D}{Dt}\int_{R(\vec{X})} f\rho J\, d^3X,$$

where

$$J = \left| \frac{\partial(x_i)}{\partial(X_i)} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}.$$

Since $\vec{X}$ is fixed in the moving fluid,

$$\frac{D}{Dt} \int_{R(\vec{x})} f\rho \, d^3x = \int_{R(\vec{X})} \frac{D}{Dt}(f\rho J) \, d^3X.$$

For conservation of mass, we take $f = 1$. Then

$$\frac{D}{Dt} \int_{R(\vec{x})} \rho \, d^3x = \int_{R(\vec{X})} \frac{D}{Dt}(\rho J) \, d^3X = 0,$$

and since R is arbitrary,

$$\frac{D}{Dt}(\rho J) = 0. \tag{6.1}$$

This is a somewhat peculiar form of the continuity equation. It is, however, easily converted to the usual form:

$$\frac{D}{Dt}(\rho J) = \rho \frac{DJ}{Dt} + J \frac{D\rho}{Dt},$$

$$\begin{aligned} \frac{1}{J} \frac{DJ}{Dt} &= \frac{\frac{\partial(u_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}}{\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}} + \frac{\frac{\partial(x_1,u_2,x_3)}{\partial(X_1,X_2,X_3)}}{\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}} + \frac{\frac{\partial(x_1,x_2,u_3)}{\partial(X_1,X_2,X_3)}}{\frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)}} \\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \nabla \cdot \vec{u} \end{aligned}$$

and

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \tag{6.2}$$

Returning to the general case

$$
\begin{aligned}
\frac{D}{Dt} \int_{R(\vec{x})} f\rho \, d^3x &= \int_{R(\vec{X})} \frac{D}{Dt}(f\rho J) \, d^3X \\
&= \int_{R(\vec{X})} \left\{ \frac{Df}{Dt}\rho J + f \underbrace{\frac{D}{Dt}(\rho J)}_{=0} \right\} d^3X \\
&= \int_{R(\vec{X})} \frac{Df}{Dt}\rho J \, d^3X \\
&= \int_{R(\vec{x})} \frac{Df}{Dt}\rho \, d^3x,
\end{aligned}
$$

that is, we can move $\frac{D}{Dt}$ inside an integral within Eulerian space. Note also, $\frac{D}{Dt}$ is applied to f *not* ρf.

6.2 Newton's second law − for fluids

Newton's second law for a volume of fluid, R, is

$$
\frac{D}{Dt} \underbrace{\int_R \rho u_i \, d^3x}_{momentum} = \underbrace{\int_R \rho f_i \, d^3x}_{body\ force}
$$

$$
+ \qquad \underbrace{\int_S \sigma_{ij} n_j \, dS,}_{force\ exerted\ on\ surface\ of\ R\ by\ fluid\ outside\ R} \tag{6.3}
$$

N.B. $\int_S \sigma_{ij} n_j \, dS$ can be rewritten $\int_S F_i \, dS$, where the surface force, $\vec{F}$, is given by $F_i = \sigma_{ij} n_j$ ($\vec{n}$ is the outward normal). The stress tensor, σ_{ij}, represents the flux of i-momentum in the minus j-direction (recall that $\vec{n}$ is the outward normal whereas we are considering the force exerted on S by the fluid outside R.). Intuitively, we expect the flux of i-momentum in the i-direction to be related to pressure.

Now,

$$\frac{D}{Dt} \int_R \rho u_i \, d^3x = \int_R \rho \frac{Du_i}{Dt} d^3x.$$

Also, by the divergence theorem,

$$\int_S \sigma_{ij} n_j \, dS = \int_R \frac{\partial \sigma_{ij}}{\partial x_j} d^3x.$$

Finally, since R is arbitrary, we have

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{6.4}$$

Note that ρ is outside the derivative. Equation 6.4 is not the usual form of the momentum equation (in particular, the pressure gradient term is buried in $\frac{\partial \sigma_{ij}}{\partial x_j}$); as our first step in evaluating $\frac{\partial \sigma_{ij}}{\partial x_j}$, we will consider *angular momentum*:

$$\frac{D}{Dt} \int_R \vec{r} \times \rho \vec{u} \, d^3x = \int_R \rho \vec{r} \times \vec{f} \, d^3x + \int_S \vec{r} \times \vec{F} \, dS, \tag{6.5}$$

where $\vec{F} = \sigma_{ij} n_j \hat{i}_i$[1]. (Note that (6.5) assumes no intrinsic torques.)
Rewriting (6.5),

$$\underbrace{\int_R \rho \frac{D}{Dt} (\vec{r} \times \vec{u}) \, d^3x}_{A} = \underbrace{\int_R \rho \vec{r} \times \vec{f} \, d^3x}_{B} + \underbrace{\int_R \hat{i}_i \frac{\partial}{\partial x_l} (\epsilon_{ijk} x_j \sigma_{kl}) \, d^3x}_{C} \text{[2]}.$$

---

[1] The quantity $\hat{i}_i$ is a unit vector in the i-direction.
[2] The quantity ϵ_{ijk} is called an alternant and is defined as follows

$$\begin{aligned} \epsilon_{ijk} &\equiv 1 \text{ for } ijk = 123,\ 231,\ 312 \\ &\equiv -1 \text{ for } ijk = 321,\ 213,\ 132 \\ &\equiv 0 \text{ when any two of } ijk \text{ are equal.} \end{aligned}$$

Similarly, note that δ_{ij} is the Kronecker Delta, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$.

$$A = \int_R \rho \left[\overbrace{\underbrace{\frac{D\vec{r}}{Dt}}_{=0} \times \vec{u}}^{\vec{u}} + \vec{r} \times \frac{D\vec{u}}{Dt} \right] d^3x$$

$$B = \int_R \vec{r} \times \underbrace{\left(\rho \frac{D\vec{u}}{Dt} - \frac{\partial \sigma_{ij}}{\partial x_j} \hat{i}_i \right)}_{=\vec{f} \text{ from Newton's second law}} d^3x$$

$$C = \int_R \hat{i}_i \left[\epsilon_{ijk}\delta_{jl}\sigma_{kl} + \epsilon_{ijk}x_j \frac{\partial \sigma_{kl}}{\partial x_l} \right] d^3x.$$

After obvious cancellation, we are left with

$$\int_R \hat{i}_i \epsilon_{ijk}\delta_{jl}\sigma_{kl} \, d^3x = 0$$

or

$$\hat{i}_i \epsilon_{ijk}\delta_{jl}\sigma_{kl} = 0$$

or

$$\hat{i}(\sigma_{32} - \sigma_{23}) + \hat{j}(\sigma_{13} - \sigma_{31}) + \hat{k}(\sigma_{12} - \sigma_{21}) = 0.$$

Thus, in the absence of intrinsic torques

$$\sigma_{ij} = \sigma_{ji}. \tag{6.6}$$

6.3   Energy

Let us first look at the rate of change of *mechanical* energy. Take the momentum equation (6.4), multiply by u_i and sum over i:

$$\rho \frac{D}{Dt}\left(\frac{u_i u_i}{2}\right) = \rho f_i u_i + u_i \frac{\partial \sigma_{ij}}{\partial x_j}. \tag{6.7}$$

Integrate (6.7) over region R

$$\overbrace{\frac{D}{Dt}\int_R \frac{1}{2}\rho u_i u_i \, d^3x}^{A} = \overbrace{\int_R \rho f_i u_i \, d^3x}^{B} + \int_R u_i \frac{\partial \sigma_{ij}}{\partial x_j} d^3x.$$

The last term can be rewritten

$$
\begin{aligned}
\int_R u_i \frac{\partial \sigma_{ij}}{\partial x_j} d^3x &= \int_R \frac{\partial}{\partial x_j}(u_i \sigma_{ij}) \, d^3x - \int_R \frac{\partial u_i}{\partial x_j}\sigma_{ij} \, d^3x \\
&= \underbrace{\int_S u_i \sigma_{ij} n_j \, dS}_{C} - \underbrace{\int_R \frac{\partial u_i}{\partial x_j}\sigma_{ij} \, d^3x}_{D}.
\end{aligned}
$$

The labelled terms are interpreted as follows:

Term A: Time rate of change of mechanical energy.

Term B: Work done by body forces.

Term C: Work done by surface stresses.

Term D: Needs elucidation!

Recall, there is no conservation of mechanical energy alone. What about total energy?

$$
\begin{aligned}
\frac{D}{Dt}\int_R \rho\left(\frac{u^2}{2}+e\right) d^3x &= \int_R \rho f_i u_i \, d^3x + \int_S u_i \sigma_{ij} n_j \, dS \\
&\quad + \int_S -\vec{K}\cdot\vec{n}\, dS + \int_R \rho Q \, d^3x
\end{aligned}
$$

($e = c_V T$, c_V = heat capacity at constant volume, Q = external heating, and $\vec{K}$ = heat flux). In view of the arbitrariness of R,

$$\rho\frac{D}{Dt}\left(\frac{u^2}{2}+e\right) = \rho f_i u_i + \frac{\partial}{\partial x_j}(\sigma_{ij}u_i) - \nabla\cdot\vec{K} + \rho Q.$$

Subtract from the above the relation for kinetic energy:

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial u_i}{\partial x_j} - \nabla \cdot \vec{K} + \rho Q. \tag{6.8}$$

Equations 6.2, 6.4, 6.6, and 6.8 are our equations of motion – so far.

6.4 $\vec{K}$ and σ_{ij}

The nature of $\vec{K}$ and σ_{ij} is usually (and properly) discussed in terms of molecular collisions and/or turbulent mixing. We will take a somewhat different approach here. Let us assume

$$\vec{K} = \vec{K} \left(\rho, T, \frac{\partial T}{\partial x_i} \right).$$

Taylor expanding in $\frac{\partial T}{\partial x_i}$ we get

$$K_i = K_i^{(0)}(\rho, T) + A_{ij}(\rho, T) \frac{\partial T}{\partial x_j} + \dots .$$

In general, $K_i^{(0)} = 0$. If we also assume transport to be *isotropic* then A_{ij} must be proportional to δ_{ij}; that is,

$$A_{ij} = -k(\rho, T)\delta_{ij},$$

and

$$\vec{K} = -k(\rho, T)\nabla T. \tag{6.9}$$

This result is, of course, far more convincingly obtained by kinetic theory considerations.

Following a similarly abstract approach for σ_{ij} we will assume

$$\sigma_{ij} = \sigma_{ij} \left(\rho, T, \frac{\partial u_k}{\partial x_l} \right).$$

(Fluids satisfying this assumption are known as *Newtonian* fluids.) Taylor expanding the above relation we get

$$\sigma_{ij} = B_{ij}(\rho, T) + C_{ijkl}(\rho, T)\frac{\partial u_k}{\partial x_l} + \dots .$$

Again we want B_{ij} and C_{ijkl} to be isotropic tensors.

$$B_{ij} = - \underbrace{p}_{pressure} \delta_{ij}$$

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \underbrace{(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})}_{symmetric\ in\ i,j} + \nu \underbrace{(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il})}_{antisymmetric\ in\ i,j}$$

Since $\sigma_{ij} = \sigma_{ji}$, $\nu \equiv 0$ and

$$\sigma_{ij} = -p\delta_{ij} + \underbrace{\lambda\delta_{ij}\nabla \cdot \vec{u} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\tau_{ij},\ the\ viscous\ stress\ tensor}, \qquad (6.10)$$

where $\mu \equiv$ first viscosity and $\lambda \equiv$ second viscosity. The expression for τ_{ij} is a little more complicated than the most common expressions. Stokes suggested that average normal viscous stress should be zero; that is, $\tau_{ii} = 3\lambda\nabla \cdot \vec{u} + 2\mu\nabla \cdot \vec{u}$, which implies $\lambda + (2/3)\mu = 0$. The quantity $\eta = \lambda + (2/3)\mu$ is called the *bulk viscosity* and is zero for spherical molecules – but not otherwise. Still, for incompressible fluids where $\nabla \cdot \vec{u} \equiv 0$, the second viscosity is irrelevant. Also, if μ and λ are constant

$$\frac{\partial}{\partial x_j}\tau_{ij} = \lambda\frac{\partial}{\partial x_i}(\nabla \cdot \vec{u}) + \mu\left(\frac{\partial^2 u_i}{\partial x_j\,\partial x_j} + \frac{\partial^2 u_j}{\partial x_i\,\partial x_j}\right)$$

or

$$\hat{i}_i\frac{\partial \tau_{ij}}{\partial x_j} = (\lambda + \mu)\nabla(\nabla \cdot \vec{u}) + \underbrace{\mu\nabla^2\vec{u}}_{most\ commonly\ considered\ term}.$$

With (6.10), $\sigma_{ij}\frac{\partial u_i}{\partial x_j}$ in (6.8) becomes

$$\sigma_{ij}\frac{\partial u_i}{\partial x_j} = -p\nabla \cdot \vec{u} + \Phi,$$

where $\Phi \equiv \tau_{ij}\frac{\partial u_i}{\partial x_j}$. Φ represents dissipation by viscosity. It can be shown that $\Phi \geq 0$ if and only if $\mu \geq 0$, $\eta \geq 0$. Using (6.9) and (6.10), and replacing $f_i \hat{i}_i$ with $\vec{g}$, (6.4) and (6.8) become

$$\rho\frac{Du_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{6.11}$$

and

$$\rho c_V \frac{DT}{Dt} + p\nabla \cdot \vec{u} = \nabla \cdot (k\nabla T) + \rho Q + \Phi. \tag{6.12}$$

Note that Φ is generally small for typical geophysical scales.

6.5   Equations of state

Equations 6.2, 6.11, and 6.12 represent five equations in six unknowns u_i, ρ, p, T. The remaining equation is the *equation of state*. Several choices are commonly made.

$$\text{Perfect gas:} \quad p = \rho R T \tag{6.13}$$
$$\text{Boussinesq fluid:} \quad \rho = \rho_0(1 - \alpha(T - T_0)) \tag{6.14}$$
$$\text{homogeneous, incompressible fluid:} \quad \rho = \text{constant.} \tag{6.15}$$

In Equation 6.12 it is common to rewrite the left-hand side as follows:

$$\rho c_V \frac{DT}{Dt} + p\nabla \cdot \vec{u} = \rho c_V \frac{DT}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = \rho T \frac{DS}{Dt},$$

where S is entropy; note that thermodynamic equilibrium has been assumed. For a perfect gas

$$\begin{aligned}
\frac{DS}{Dt} &= \frac{c_v}{T}\frac{DT}{Dt} - \frac{R}{\rho}\frac{D\rho}{Dt} \\
&= \frac{D}{Dt}\ln\left(\frac{T^{c_v}}{\rho^R}\right). \tag{6.16}
\end{aligned}$$

The relation between S and *potential temperature* ($\Theta \equiv T^{c_V/R}/\rho$) is obvious. For the Boussinesq fluid, S is proportional to T (or ρ), while for an incompressible, homogeneous fluid S is irrelevant.

6.6 Rotating coordinate frame

Consider a rotating frame (x', y', z'). A given position $\vec{r}$ can be represented

$$
\begin{aligned}
\vec{r} &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} && \text{(inertial frame)} \\
&= x'\hat{i}' + y'\hat{j}' + z'\hat{k}' && \text{(rotating frame).}
\end{aligned}
$$

The velocity in the inertial frame can be written

$$
\begin{aligned}
\vec{u} &= \frac{D\vec{r}}{Dt} \\
&= \underbrace{\frac{Dx'}{Dt}\hat{i}' + \frac{Dy'}{Dt}\hat{j}' + \frac{Dz'}{Dt}\hat{k}'}_{velocity\ in\ rotating\ frame} \\
&\quad + \underbrace{x'\frac{D\hat{i}'}{Dt} + y'\frac{D\hat{j}'}{Dt} + z'\frac{D\hat{k}'}{Dt}}_{velocity\ of\ particle\ fixed\ in\ rotating\ frame:\ \vec{\omega}\times\vec{r}}.
\end{aligned}
$$

So,

$$
\vec{u} = \vec{u}' + \vec{\omega} \times \vec{r}
$$

and

$$
\frac{D}{Dt} = \left(\frac{D}{Dt}\right)_{rot} + \vec{\omega} \times \ .
$$

Thus for

$$
\vec{u} = \vec{u}' + \vec{\omega} \times \vec{r},
$$

$$
\begin{aligned}
\frac{D\vec{u}}{Dt} &= \left(\frac{D\vec{u}}{Dt}\right)_{rot} + \vec{\omega} \times \frac{D\vec{r}}{Dt} \\
&= \left(\frac{D}{Dt}\right)_{rot}(\vec{u}' + \vec{\omega} \times \vec{r}) + \omega \times (\vec{u}' + \vec{\omega} \times \vec{r}) \\
&= \left(\frac{D\vec{u}'}{Dt}\right)_{rot} + \vec{\omega} \times \vec{u}' + \vec{\omega} \times \vec{u}' + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \ .
\end{aligned}
$$

Now,

$$
\vec{\omega} \times (\vec{\omega} \times \vec{r}) = (\vec{\omega} \cdot \vec{r})\vec{\omega} - \omega^2\vec{r} \ = \ -\omega^2(\vec{r} - \frac{\vec{\omega}(\vec{\omega} \cdot \vec{r})}{\omega^2}) = -\omega^2\vec{R}
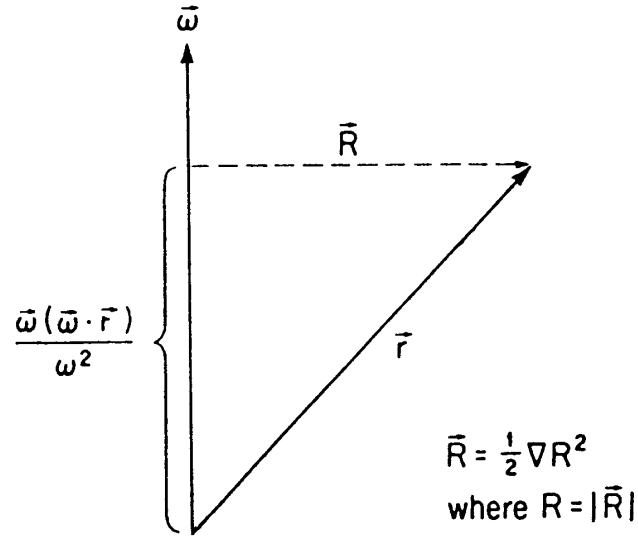$$
$$
(viz \ \text{Figure 6.1}).
$$

Figure 6.1: Centrifugal contribution to geopotential.

Thus, in a rotating frame, (6.11) becomes

$$\rho \left(\frac{D\vec{u}'}{Dt} + \overbrace{2\vec{\omega} \times \vec{u}'}^{Coriolis\ force} \right) = -\nabla p + \nabla \cdot \vec{\tau}$$

$$+ \underbrace{\rho \vec{g} + \rho \nabla \left(\frac{\omega^2 R^2}{2} \right)}_{usually\ combined\ as\ -\rho \nabla \Omega} \ . \qquad (6.17)$$

where $\Omega \equiv$ *geopotential*. Note that, because of isotropy, the viscous term is the same in rotating and non-rotating systems.

6.7 Spherical coordinates

We shall not derive the spherical equations here. The task is straightforward. The most direct approach involves transforming our equations into vector-invariant form (note that $\vec{u} \cdot \nabla$ is *not* vector-invariant), and then employing spherical forms for the invariant operations. Here we will merely write down equations discussed in Holton (his section 2.3; N.B. Holton uses an approach different from what we have just described). Holton considers a quasi-cartesian system on the surface of the earth (see Figure 6.2).

Sphere	Cartesian tangent plane	
$r = a + z$	dz	$w = \frac{dz}{dt}$
λ	$dx = a \cos \phi\, d\lambda$	$u = a \cos \phi \frac{d\lambda}{dt}$
ϕ	$dy = a\, d\phi$	$v = a \frac{d\phi}{dt}$

If we define

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z},$$
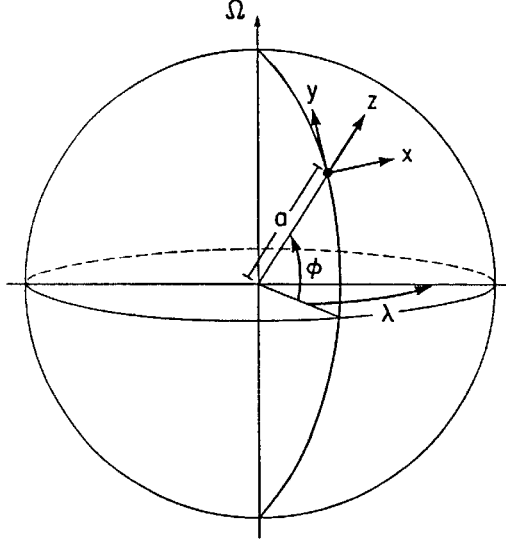
then the equations for $u, v,$ and w become

Figure 6.2: Spherical coordinates with cartesian tangent plane.

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi$$

$$-2\Omega w\cos\phi + \frac{1}{\rho}(\nabla\cdot\tau)_x \qquad (6.18)$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \frac{1}{\rho}(\nabla\cdot\tau)_y \qquad (6.19)$$

$$\frac{dw}{dt} + \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + \frac{1}{\rho}(\nabla\cdot\tau)_z \; . \qquad (6.20)$$

(What about equations for energy, continuity, and the gas law?)

6.8   Scaling

Scaling is an approach to non-dimensionalization which ideally permits ordering terms in an equation according to their size. For any variable, f, we

may write

$$f = F\tilde{f},$$

where

$$
\begin{aligned}
F &= \text{characteristic magnitude} \\
\tilde{f} &= O(1) \text{ non-dimensional quantity} \,.
\end{aligned}
$$

Such a redefinition leads to non-dimensional parameters which indicate the relative importance of various terms in equations. Some of the more famous non-dimensional numbers, and the physical balances they represent, are

$$
\begin{aligned}
\text{Rossby No.:} \quad Ro &= \frac{U}{2\Omega L} \sim \frac{\text{inertia}}{\text{Coriolis force}} \\
\text{Froude No.:} \quad Fr &= \frac{gL}{U^2} \sim \frac{\text{gravity}}{\text{inertia}} \\
\text{Reynolds No.:} \quad Re &= \frac{LU}{\nu} \sim \frac{\text{inertia}}{\text{friction}} \\
\text{Ekman No.:} \quad Ek &= \frac{Ro}{Re} = \frac{\nu}{2\Omega L^2} \sim \frac{\text{friction}}{\text{Coriolis force}} \,,
\end{aligned}
$$

(where $\nu \equiv \mu/\rho$). Note that there is little sense in scaling pressure in the absence of either data or the size of other numbers.

The following are some serious problems with scaling:

1. Northerly, westerly, and vertical scales and velocities are not necessarily the same.

2. Several scales for a single variable may be present in a single problem.

3. Not all variables are measured.

Ultimately, scaling actually requires that we already have knowledge of the answer! The following sections deal with some specific examples of scaling.

6.9   Hydrostaticity

As already noted, *hydrostaticity* is, in general, an approximation. We will use scaling to estimate the conditions needed for hydrostaticity to be a good approximation. One is tempted to note that g turns out to be much larger than any acceleration or stress in Equation 6.20 – at least for most meteorological and oceanographic systems. Thus, one might argue that the only term left to balance g is $\frac{1}{\rho}\frac{\partial p}{\partial z}$. This would, however, be a possibly misleading statement. To see why, let us write

$$
\begin{aligned}
p &= p_0(z) + p'(x, y, z, t) \\
\rho &= \rho_0(z) + \rho'(x, y, z, t) \ .
\end{aligned}
$$

The quantities p_0 and ρ_0 represent horizontal and time averages; the primed quantities represent deviations from these averages. In general, the averaged quantities dominate the above expressions; however, for stable stratification[3], the averaged terms are not directly involved in the dynamics (which depends on gradients of pressure, density, etc.). Scaling must be done *after* p_0 and ρ_0 (and T_0) have been subtracted from p and ρ (and T). The evaluation of the term $\rho'g$ in Equation 6.20 then requires knowing how ρ' is related to w, and so forth. It is here that one has to involve the energy equation in the scaling, and one discovers that a necessary condition for hydrostaticity to hold is that $N\tau \gg 1$ (where N is the *Brunt-Vaisala* frequency and τ is a characteristic time scale).

   This is most easily seen for a linearized adiabatic Boussinesq fluid where (6.20) becomes

$$
\rho\frac{\partial w}{\partial t} \sim -g\rho - \frac{\partial p}{\partial z}. \tag{6.21}
$$

Now, $\rho = \rho_0 + \rho'$, and $\frac{\partial p_0}{\partial z} = -g\rho_0$. Subtracting these from (6.21), we get for our vertical momentum equation

$$
\rho_0\frac{\partial w}{\partial t} \sim -g\rho' - \frac{\partial p'}{\partial z} \ .
$$

---

[3]We'll explain what this means soon.

Similarly, the Boussinesq energy equation becomes

$$\frac{\partial \rho'}{\partial t} + w\frac{d\rho_0}{dz} \sim 0.$$

For convenience, let our variables have the following time dependence: $e^{i\omega t}$. The last two equations become

$$\rho_0 i\omega w \sim -g\rho' - \frac{\partial p'}{\partial z}$$

$$i\omega \rho' \sim -w\frac{d\rho_0}{dz}.$$

Eliminating w from the above equations yields

$$\left(\frac{\rho_0 \omega^2}{\frac{d\rho_0}{dz}}\right)\rho' \sim -g\rho' - \frac{\partial p'}{\partial z},$$

from which we see that hydrostaticity requires that

$$\left|\frac{\rho_0 \omega^2}{g\frac{d\rho_0}{dz}}\right| \ll 1,$$

or

$$\left|\frac{\omega^2}{N^2}\right| \ll 1,$$

where $N^2 \equiv -\frac{g}{\rho_0}\frac{d\rho_0}{dz}$. The quantity N is called the Brunt-Vaisala frequency and is a measure of the stratification. When $N^2 < 0$, the fluid is statically unstable (that is, we have heavier fluid on top of lighter fluid)[4].

---

[4]We will have much more to say about both the Brunt-Vaisala frequency and static stability in later chapters.

6.10   Geostrophy

In terms of scaling, one commonly finds that $Ro \ll 1$ and $Ek \ll 1$ so that pressure gradients primarily balance the Coriolis force. We also saw this directly in the data. Nevertheless, the existence of this approximate balance, alone, does not permit us to calculate the time evolution of flow fields. In order to exploit $Ro \ll 1$ to calculate the time evolution of almost geostrophic fields we will have to perform a more sophisticated scaling analysis (not too different from what we just did in connection with hydrostaticity) which leads to what are called the *quasi-geostrophic equations*. We will defer this analysis until a later chapter.

Before ending this chapter, an important aspect of geostrophic balance needs to be mentioned and pondered: namely, in the absence of viscosity, thermal conductivity, and x-variation, the following geostrophic flow is an exact solution of the nonlinear equations:

$$
\begin{aligned}
T(y,z) \quad & \text{as determined from } Q = 0; \text{ that is, radiative equilibrium;} \\
p, \rho \quad & \text{as determined from hydrostaticity and the gas law;} \\
u \quad & \text{as determined from geostrophy; and} \\
v = w \quad & = 0 \ .
\end{aligned}
$$

The intrinsic difficulties with the above solution will form the focus for much of our discussion in Chapter 7.