

# Chapter 13

## The generation of eddies by instability

### Supplemental reading:

Holton (1979), sections 9.2, 9.3

Pedlosky (1979), sections 7.1–3

### 13.1 Remarks

In the previous chapters, we examined the wave properties of the atmosphere under a variety of circumstances. We have also considered the interactions of waves with mean flows. Within General Circulation Theory, it is generally held that eddies have to be involved in transporting heat between the tropics and the poles. Thus far our study of waves has not provided much insight into this matter. As it turns out, vertically propagating stationary Rossby waves do carry heat poleward. This heat transport, while not insignificant, is largely a byproduct of the fact that the wave momentum flux acts to reduce shears, and geostrophic adjustment involves a concomitant reduction of meridional temperature gradients. Unfortunately, we will not have time to study these mechanisms in these notes. However, the quasi-geostrophic framework established in Chapter 5 greatly simplifies such studies. This framework will be used to study how travelling disturbances arise in the atmosphere. We will also sketch some results which suggest that these travelling disturbances play the major role in determining the global, temperature distribution. The

generation of such disturbances involves hydrodynamic instability, and before diving into this problem in a meteorological context, it will be useful to examine stability in simpler situations.

Before beginning this topic we should recall possible wave (eddy) sources considered thus far:

1. Direct forcing as produced by tidal heating, flow over mountains, or flow through quasi-stationary inhomogeneities in heating.
2. Resonant free oscillations. These are presumably preferred responses to any ‘noise’. However, given the presence of dissipation it is not clear what precisely maintains the free Rossby waves observed in the atmosphere. Moreover, when the relative phase speeds of Rossby waves become small compared to *variations* in the mean zonal flow, the free Rossby waves cease to exist. Observed free Rossby waves have large phase speeds.

The bulk of the *travelling* disturbances in the (lower) atmosphere and oceans are due to neither of the above, but arise as instabilities on the mean flow.

## 13.2 Instability

We shall use the word instability to refer to any situation where a perturbation extracts ‘energy’ from the unperturbed flow. The word ‘energy’ is surrounded by quotes because the concept of energy is not always unambiguous. Crudely, an instability grows at the expense of the basic flow. The precise sense in which this is true may have to be elaborated on.

This topic has been studied for well over a century. It is still a major area of research with many areas of uncertainty and ignorance. In this chapter, we can only hope to convey a taste of what is an important, interesting, and difficult subject.

The most commonly studied approach to instability is by way of what are called ‘normal mode’ instabilities. We earlier referred to the solutions of the homogeneous perturbation equations as free oscillations. These were normal mode solutions in the sense that an initial perturbation of such form would continue in that form; this would not be true for an arbitrary (or non-normal mode) initial perturbation. In particularly simple situations, we

solved for the frequencies of these oscillations. In the situations we studied ( $U_0 = \text{constant}$ ) these frequencies were real, but in other situations these frequencies can have an imaginary part,  $\sigma_i$ . When the sign of  $\sigma_i$  is such as to imply exponential growth in time, the basic state is said to be unstable with respect to normal mode perturbations.

Most of our attention will be devoted to these normal mode instabilities, but you should be aware that these are not the only cases of instability. It frequently occurs that arbitrary initial perturbations have algebraic rather than exponential growth. In addition disturbances may have initial algebraic growth followed by algebraic decay. Such situations have frequently been ignored in the past because of the eventual decay, but clearly temporary growth – especially when rapid – may be of considerable practical importance. The plethora of possibilities may be a little confusing, but it is important to be aware of them. We will present examples of non-normal mode instability later in this chapter.

### 13.2.1 Buoyant convection

This particular example is chosen as a simple example of the traditional normal mode approach to instability. The mathematical basis for our inquiry is the treatment of simple internal gravity waves in a Boussinesq fluid given in Chapter 2. Recall that we were looking at two-dimensional perturbations in the  $x, z$ -plane on a static basic state with Brunt-Vaisala frequency  $N$ . The perturbation vertical velocity,  $w$ , satisfied the following equation:

$$w_{zz} + \left\{ \left( \frac{N^2}{\sigma^2} - 1 \right) k^2 \right\} w = 0, \quad (13.1)$$

where  $w$  had an  $x, t$  dependence of the form  $e^{i(kx - \sigma t)}$ . As boundary conditions we will take

$$w = 0 \text{ at } z = 0, H. \quad (13.2)$$

Thus far we have introduced nothing new. However, we will now take  $N^2 < 0$ ! As in our earlier analysis, (13.1) has solutions of the form

$$w = \sin \lambda z, \quad (13.3)$$

where

$$\lambda = \left( \frac{N^2}{\sigma^2} - 1 \right)^{1/2} k = \frac{n\pi}{H}, \quad n = 1, 2, \dots \quad (13.4)$$

Solving for  $\sigma^2$  we again get

$$\sigma^2 = \frac{N^2}{1 + \frac{n^2\pi^2}{k^2H^2}}, \quad (13.5)$$

but now  $\sigma^2 < 0$ , and

$$\sigma = \pm i \left\{ \frac{-N^2}{1 + \frac{n^2\pi^2}{k^2H^2}} \right\}^{1/2}. \quad (13.6)$$

The largest growth rate is associated with  $n = 1$  and  $k = \infty$ , and is given by

$$\sigma = i(-N^2)^{1/2}. \quad (13.7)$$

### 13.2.2 Rayleigh–Benard instability

In reality viscosity and thermal conductivity tend to suppress small scales. Very crudely, they produce a damping rate

$$d \sim \nu(\lambda^2 + k^2) \quad (13.8)$$

so that (13.6) becomes

$$\sigma \approx i \left\{ \underbrace{\left\{ \frac{-N^2}{1 + \frac{n^2\pi^2}{k^2H^2}} \right\}^{1/2}}_A - \underbrace{\nu k^2 \left( 1 + \frac{n^2\pi^2}{k^2H^2} \right)}_B \right\}. \quad (13.9)$$

Term A exceeds term B only over a finite range of  $k$ , provided  $-N^2$  is large enough. A schematic plot of A and B illustrates this. As an exercise you will work out the critical value of  $N^2$  and the optimum  $k$ . The above problem is usually referred to as *Rayleigh-Benard* instability for stress-free boundaries. In the atmosphere and oceans, even slightly unstable conditions lead to large values of  $-N^2$ ; maximum growth rates very nearly approach the value given by (13.7) for

$$k \sim O\left(\frac{1}{H}\right).$$

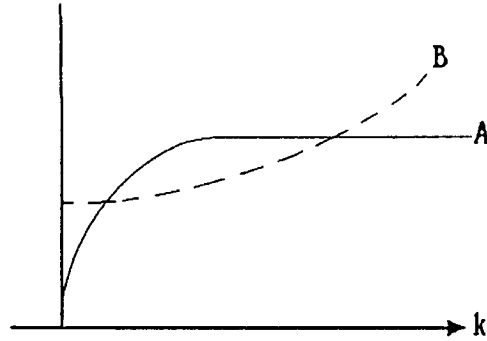


Figure 13.1: A plot of the buoyant growth term,  $A$ , and the diffusive damping term,  $B$ , in Equation 13.9.

### 13.2.3 Convective adjustment and gravity wave breaking

Such growth times are so much shorter than typical time scales for medium–large scale motions as to imply that convection will prevent  $N^2 \lesssim 0$  over the longer time scales (at least above the surface boundary layer). This process of *convective adjustment* is of substantial importance in atmospheric and oceanic physics and in modelling.

A practical application of convection adjustment arises in connection with vertically propagating gravity waves. Recall that such waves increase in amplitude as  $e^{z^*/2}$  and also that such waves are approximately solutions to the nonlinear equations. Thus at some height the temperature field associated with the wave should become statically unstable – but for the onset of convection. The intensity of the convection (turbulence) ought to be proportional to the time rate of change in temperature which the wave would produce in the absence of turbulence. This mechanism is currently believed to account for the turbulence in the mesosphere called for in composition calculations.

### 13.2.4 Reversal of mesopause temperature gradient

The ‘wave breaking’ also must lead to the deposition of the waves’ momentum flux in the mean flow. If the waves originate in the troposphere (and thus have small values of phase speed  $c$ ), then the deposition of their momentum flux will lead to a slowing of mesospheric zonal winds and the reversal of the pole–pole temperature gradient as observed at the mesopause (Why?).

### 13.2.5 Kelvin-Helmholtz instability

This problem consists in the investigation of the free solutions in a stratified (constant  $N^2$ ), non-rotating, infinite Boussinesq fluid with the following basic velocity profile:

$$\begin{aligned} u_0 &= U \text{ for } z > 0 \\ u_0 &= -U \text{ for } z < 0. \end{aligned} \quad (13.10)$$

Away from  $z = 0$ , solutions of the form  $e^{ik(x-ct)}$  satisfy

$$\frac{d^2 w'}{dz^2} + \left[ \frac{N^2}{(u_0 - c)^2} - k^2 \right] w' = 0. \quad (13.11)$$

For an infinite fluid our boundary conditions are that  $w$  remain bounded as  $|z| \rightarrow \infty$ . Also, if

$$\frac{N^2}{(u_0 - c)^2} - k^2$$

is positive for either  $z > 0$  or  $z < 0$ , we require the appropriate radiation condition.

The discontinuity in  $u_0$  at  $z = 0$  means that we have different solutions for  $z > 0$  and  $z < 0$ . At  $z = 0$  we require continuity of perturbation vertical displacement and pressure. It is easy to show that

$$\frac{p'}{\rho_0} = \frac{u_0 - c}{ik} \frac{dw'}{dz} \quad (13.12)$$

and

$$w' = ik(u_0 - c)Z, \quad (13.13)$$

where  $Z =$  vertical displacement.

Thus we require continuity of

$$(u_0 - c) \frac{dw'}{dz}$$

and

$$\frac{w'}{(u_0 - c)} \quad (13.14)$$

at  $z = 0$ .

It is also easy to show that no free solutions exist for  $|\operatorname{Re}(c)| > U$ . (Why? Hint: Use Eliassen-Palm theorems.)

Now let

$$n_1^2 = \left( \frac{N^2}{(U - c)^2} - k^2 \right) \quad (13.15)$$

$$n_2^2 = \left( \frac{N^2}{(U + c)^2} - k^2 \right). \quad (13.16)$$

Then for  $z > 0$

$$w' = A_1 e^{\pm i n_1 z}, \quad (13.17)$$

and for  $z < 0$

$$w' = A_2 e^{\pm i n_2 z}. \quad (13.18)$$

The choice of sign in (13.17) and (13.18) is made to satisfy boundary conditions as  $z \rightarrow \pm\infty$ . Once these choices are made, (13.14) yields the relation between  $c$  and  $k$ . Solving for this calls for a fair amount of algebra which can be found in Lindzen (1974) and Lindzen and Rosenthal (1976). Here we shall merely cite the results.

### 13.2.6 Radiating and growing solutions

(i) For  $k < \frac{N}{U}$  we have a solution where  $c_i = c_r = 0$ ,

$$\begin{aligned} w &= Ae^{in_1z} \text{ for } z > 0 \\ &= -Ae^{in_2z} \text{ for } z < 0. \end{aligned} \quad (13.19)$$

(ii) For  $\frac{N}{2U} < k < \frac{N}{\sqrt{2}U}$  we also have solutions given by (13.19) where  $c_i = 0$  and

$$c_r = \pm \left[ \frac{N^2}{2k^2} - U^2 \right]^{1/2}. \quad (13.20)$$

(iii) For  $k > \frac{N}{\sqrt{2}U}$  we have solutions for which  $c_r = 0$  and

$$c_i = \left( U^2 - \frac{N^2}{2k^2} \right)^{1/2}, \quad (13.21)$$

where

$$\begin{aligned} w &= A_1e^{-nz} \text{ for } z > 0 \\ &= A_2e^{n^*z} \text{ for } z < 0 \end{aligned} \quad (13.22)$$

and

$$n^2 = -\frac{k^2(U + ic_i)^2}{(U - ic_i)^2}, \quad (13.23)$$

where  $n =$  that root of (13.23) with a positive real part. Also,

$$A_2 = -\frac{U + ic_i}{U - ic_i}A_1. \quad (13.24)$$

The above results show that a strong shear zone can generate both growing interfacial disturbances confined to the shear zone and internal gravity waves propagating away from the shear zone. In each case the real phase speeds are confined between  $\pm U$ .

The instabilities are known as Kelvin-Helmholtz waves. Both these and the radiating gravity waves appear to play a major role in clear air turbulence.



The radiating gravity waves are a manifestation of the mean flow giving up energy to eddies. In that sense, they also constitute a type of instability.

This becomes clearer if we consider the solution consisting in gravity waves radiating away from the shear layer a little further. When waves radiate away from a region without any waves approaching that region, we have what amounts to an infinite reflection coefficient. This is an extreme example of over-reflection. Over-reflection refers to situations where waves are reflected with reflection coefficients that exceed one. You may confirm for yourself in the present problem that gravity waves incident on the shear layer with phase speeds between  $\pm U$  will, in general, be over-reflected. Finally, it should be noted that if we had a reflecting boundary below and/or above the shear layer, then over-reflection could lead to growing modes. A wave approaching the shear layer would be over-reflected and returned to the reflecting boundary with increased amplitude. Reflection at the boundary would return the wave to the over-reflecting shear layer for further amplification. Such a process could obviously lead to continuous magnification – provided that reflected and over-reflected waves were in phase so as to avoid destructive interference. This is described in detail in Lindzen and Rosenthal (1976).

Before ending this section, a comment is in order on the Helmholtz velocity profile. The discontinuity in  $U$  at  $z = 0$  is easily handled by the matching conditions (Equation 13.14). However, this ease tends to obscure the fact that the velocity discontinuity in the basic state leads to a *pair* of delta function contributions to  $d^2U_0/dz^{*2}$  at  $z = 0$ , and associated contributions to the underlying full equation for perturbations (i.e., Equation 10.8). In the next section, we will be concerned with issues like changes in sign for  $d^2U_0/dz^{*2}$ . It will prove essential to keep in mind the delta function contributions that have been effectively disguised in the present treatment.

### 13.3 Instability of meteorological disturbances; baroclinic and barotropic instability

Now that we have some idea of the formal approach to linear instability theory, we will look at the rather difficult question of whether instability can account for the travelling disturbances we saw on weather maps. Naturally,

our approach will not be comprehensive, but I will attempt to deal with a few aspects which I believe to be particularly central.

Our starting point in the theoretical analysis of this problem will be the quasi-geostrophic equations of Chapter 12. We will consider a basic state consisting of a purely zonal flow  $\bar{u}(y, z)$ ,

$$\bar{u}(y, z) = \frac{-1}{f_0} \frac{\partial \bar{\Phi}}{\partial y} \quad (13.25)$$

$$\bar{\Phi} = \bar{\Phi}(y, z) \quad (13.26)$$

$$\bar{v} \equiv 0.$$

If we write  $\Phi = \bar{\Phi} + \bar{\Phi}'$  (also  $u = \bar{u} + u', v = v', w = w'$ ), (12.57) (or (12.56)) upon linearization becomes

$$\left( \frac{\partial}{\partial t} + \bar{u}_0 \frac{\partial}{\partial x} \right) q' + \underbrace{\frac{1}{f_0} \frac{\partial \Phi'}{\partial x}}_{v'} \bar{q}_y = 0, \quad (13.27)$$

where

$$q = \frac{1}{f_0} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) + f + e^{z^*} \frac{\partial}{\partial z^*} \left( \frac{f_0}{S} e^{-z^*} \frac{\partial \Phi}{\partial z^*} \right) \quad (13.28)$$

$$\bar{q} = \frac{1}{f_0} \frac{\partial^2 \bar{\Phi}}{\partial y^2} + f + e^{z^*} \frac{\partial}{\partial z^*} \left( \frac{f_0}{S} e^{-z^*} \frac{\partial \bar{\Phi}}{\partial z^*} \right) \quad (13.29)$$

$$\frac{\partial \bar{q}}{\partial y} = -\frac{\partial^2 \bar{u}}{\partial y^2} + \beta - e^{z^*} \frac{\partial}{\partial z^*} \left( \frac{f_0^2}{S} e^{-z^*} \frac{\partial \bar{u}}{\partial z^*} \right) \quad (13.30)$$

$$q' = \frac{1}{f_0} \left( \frac{\partial^2 \Phi'}{\partial x^2} + \frac{\partial^2 \Phi'}{\partial y^2} \right) + e^{z^*} \frac{\partial}{\partial z^*} \left( \frac{f_0}{S} e^{-z^*} \frac{\partial \Phi'}{\partial z^*} \right). \quad (13.31)$$

Traditionally, one has also made the following approximations

$$H = \text{constant} \quad (13.32)$$

$$z = H z^* \quad (13.33)$$

$$w = Hw*, \quad (13.34)$$

in which case (13.30) becomes

$$\begin{aligned} \frac{\partial \bar{q}}{\partial y} &= -\frac{\partial^2 \bar{u}}{\partial y^2} + \beta - e^{z/H} H \frac{\partial}{\partial z} \left( \frac{f_0}{S} e^{-z/H} H \frac{\partial \bar{u}}{\partial z} \right) \\ &= -\frac{\partial^2 \bar{u}}{\partial y^2} + \beta - e^{z/H} \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} e^{-z/H} \frac{\partial \bar{u}}{\partial z} \right) \\ &= \underbrace{-\frac{\partial^2 \bar{u}}{\partial y^2} + \beta + \frac{1}{H} \frac{f_0^2}{N^2} \frac{\partial \bar{u}}{\partial z} - \frac{f_0^2}{N^2} \frac{\partial^2 \bar{u}}{\partial z^2}}_{\text{assuming } N^2 \text{ is independent of } z}. \end{aligned} \quad (13.35)$$

Note

$$\frac{H^2}{S} = \frac{H^2}{RH \left( \frac{\partial T_0}{\partial z} + \frac{g}{c_p} \right)} = \frac{1}{\left( \frac{g}{T_0} \left( \frac{dT_0}{dz} + \frac{g}{c_p} \right) \right)} = \frac{1}{N^2}.$$

Also,

$$q' = \frac{1}{f_0} (\nabla_H^2 \Phi') + e^{z/H} \frac{\partial}{\partial z} \left( \frac{f_0}{N^2} e^{-z/H} \frac{\partial \Phi'}{\partial z} \right). \quad (13.36)$$

Consistent with this approximation, (12.52) becomes

$$\left( \frac{\partial}{\partial t} + u_G \frac{\partial}{\partial x} + v_G \frac{\partial}{\partial y} \right) \frac{\partial \Phi}{\partial z} + w N^2 = 0, \quad (13.37)$$

which becomes, on linearization,

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \Phi'}{\partial z} - \frac{\partial \Phi'}{\partial x} \frac{\partial \bar{u}}{\partial z} + w' N^2 = 0. \quad (13.38)$$

To summarize, the equation for  $\Phi'$  that we obtain from (13.27) (using (13.36) and (13.34)) is:

$$\begin{aligned} \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \left( \nabla_H^2 \Phi' + e^{z/H} \frac{\partial}{\partial z} \left( \epsilon e^{-z/H} \frac{\partial \Phi'}{\partial z} \right) \right) \\ + \frac{\partial \Phi'}{\partial x} \left( \beta - \frac{\partial^2 \bar{u}}{\partial y^2} - e^{z/H} \frac{\partial}{\partial z} \left( \epsilon e^{-z/H} \frac{\partial \bar{u}}{\partial z} \right) \right) = 0, \end{aligned} \quad (13.39)$$

where

$$\epsilon \equiv f_0^2/N^2.$$

Our lower boundary condition

$$w' = 0 \text{ at } z = 0 \quad (13.40)$$

becomes (using (13.38))

$$\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \Phi'}{\partial z} - \frac{\partial \Phi'}{\partial x} \frac{\partial \bar{u}}{\partial z} = 0 \text{ at } z = 0. \quad (13.41)$$

As an upper boundary condition we either assume (13.41) to hold at some upper lid or require suitable boundedness (or the radiation condition) as  $z \rightarrow \infty$ .

Finally, we restrict ourselves to plane wave solutions of the form  $e^{ik(x-ct)}$  (recall that for instability  $c$  must have a positive imaginary part) so that (13.39) and (13.41) become

$$(\bar{u} - c) \left( \frac{\partial^2 \Phi'}{\partial y^2} - k^2 \Phi' + e^{z/H} \frac{\partial}{\partial z} \left( \epsilon e^{-z/H} \frac{\partial \Phi'}{\partial z} \right) \right) + \Phi' \bar{q}_y = 0 \quad (13.42)$$

and

$$(\bar{u} - c) \frac{\partial \Phi'}{\partial z} - \Phi' \frac{\partial \bar{u}}{\partial z} = 0 \text{ at } z = 0. \quad (13.43)$$

Equations 13.42 and 13.43, although linear, are still very hard to solve. Indeed, with few exceptions, only numerical solutions exist. Needless to say, we shall not solve (13.42) and (13.43) here. We shall, however, establish some general properties of the solutions, and find one very simple solution.

### 13.3.1 A necessary condition for instability

Those readers who have studied fluid mechanics are likely to be familiar with *Rayleigh's inflection point theorem*. This theorem states that a necessary condition for the instability of plane parallel flow  $\bar{u}(y)$  in a non-rotating, unstratified fluid is that  $\frac{d^2 \bar{u}}{dy^2}$  change sign somewhere in the fluid. Now  $\frac{d^2 \bar{u}}{dy^2}$  is simply  $\bar{q}_y$  in such a fluid. Kuo (1949) extended this result to a rotating

barotropic fluid (essentially a fluid where horizontal velocity is independent of height) and showed that  $\beta - \frac{d^2\bar{u}}{dy^2}$  must change sign. A far more general result concerning  $\bar{q}_y$  was obtained from (13.42) and the boundary conditions by Charney and Stern (1962). The afore-mentioned results turn out to be special cases of the more general result. We will proceed to derive Charney and Stern's result.

As usual, we will assume a channel geometry where  $\Phi' = 0$  at  $y = y_1, y_2$ . The formal derivation of our condition is quite simple. We divide (13.42) by  $(\bar{u} - c)$ , multiply it by  $e^{-z/H}\Phi'^*$  ( $\Phi'^*$  is the complex conjugate of  $\Phi'$ ), and integrate over the whole  $y, z$  domain:

$$\begin{aligned}
 I &\equiv \int_0^\infty \int_{y_1}^{y_2} e^{-z/H} \Phi'^* \left\{ \frac{\partial^2 \Phi'}{\partial y^2} - k^2 \Phi' + e^{z/H} \frac{\partial}{\partial z} \left( \epsilon e^{-z/H} \frac{\partial \Phi'}{\partial z} \right) \right. \\
 &\qquad \qquad \qquad \left. + \Phi' \frac{\bar{q}_y}{\bar{u} - c} \right\} dy dz \\
 &= \int_0^\infty \int_{y_1}^{y_2} e^{-z/H} \underbrace{\left\{ \frac{\partial}{\partial y} \left( \Phi'^* \frac{\partial \Phi'}{\partial y} \right) - \frac{\partial \Phi'^*}{\partial y} \frac{\partial \Phi'}{\partial y} - k^2 \Phi'^* \Phi' \right\}}_{\text{integrates to zero}} \\
 &\qquad \qquad \qquad + \Phi'^* \Phi' \frac{\bar{q}_y}{\bar{u} - c} \Big\} dy dz \\
 &+ \int_0^\infty \int_{y_1}^{y_2} \underbrace{\left\{ \frac{\partial}{\partial z} \left( \epsilon e^{-z/H} \Phi'^* \frac{\partial \Phi'}{\partial z} \right) - \epsilon e^{-z/H} \frac{\partial \Phi'^*}{\partial z} \frac{\partial \Phi'}{\partial z} \right\}}_A dy dz \\
 &= 0.
 \end{aligned}$$

Using (13.43), the integral of term  $A$  above can be rewritten

$$\int_0^\infty \int_{y_1}^{y_2} A dy dz = - \int_{y_1}^{y_2} \epsilon \Phi'^* \frac{\partial \Phi'}{\partial z} dy \Big|_{z=0}$$

$$= - \int_{y_1}^{y_2} \epsilon \Phi'^* \Phi' \frac{\partial \bar{u} / \partial z}{\bar{u} - c} dy \Big|_{z=0}.$$

We then obtain

$$\begin{aligned} I &= - \int_0^\infty \int_{y_1}^{y_2} e^{-z/H} \left\{ \left| \frac{\partial \Phi'}{\partial y} \right|^2 + k^2 |\Phi'|^2 + \epsilon \left| \frac{\partial \Phi'}{\partial z} \right|^2 \right\} dy dz \\ &\quad - \int_{y_1}^{y_2} \epsilon |\Phi'|^2 \frac{\partial \bar{u} / \partial z}{\bar{u} - c} dy \Big|_{z=0} \\ &\quad + \int_0^\infty \int_{y_1}^{y_2} e^{-z/H} |\Phi'|^2 \frac{\bar{q}_y}{\bar{u} - c} dy dz = 0. \end{aligned} \tag{13.44}$$

Now the real and imaginary parts of (13.44) must each equal zero. The imaginary part arises from the last two terms when  $c$  is complex.

Let

$$c = c_r + ic_i.$$

Then

$$\frac{1}{\bar{u} - c} = \frac{1}{\bar{u} - c_r - ic_i} = \frac{\bar{u} - c_r + ic_i}{|\bar{u} - c|^2}.$$

Also let

$$P = \frac{e^{-z/H} |\Phi'|^2}{|\bar{u} - c|^2}.$$

The imaginary part of (13.44) becomes

$$c_i \left\{ - \int_{y_1}^{y_2} \epsilon P \frac{\partial \bar{u}}{\partial z} dy \Big|_{z=0} + \int_0^\infty \int_{y_1}^{y_2} P \frac{\partial \bar{q}}{\partial y} dy dz \right\} = 0. \tag{13.45}$$

Next let us define

$$\tilde{q} = \bar{q} + \epsilon \frac{1}{f_0} \frac{\partial \bar{\Phi}}{\partial z} \delta(z - 0+), \tag{13.46}$$

where  $\delta$  is the Dirac delta function. Then (13.45) becomes

$$c_i \int_0^\infty \int_{y_1}^{y_2} P \frac{\partial \tilde{q}}{\partial y} dy dz = 0. \quad (13.47)$$

If  $c_i \neq 0$ ,

$$\int_0^\infty \int_{y_1}^{y_2} P \frac{\partial \tilde{q}}{\partial y} dy dz = 0. \quad (13.48)$$

But  $P$  is positive definite; therefore, there must be some surface (possibly  $z = 0+$ ) where  $\frac{\partial \tilde{q}}{\partial y}$  changes sign. When  $\bar{u} = \bar{u}(y)$ , this reduces to  $\beta - \bar{u}_{yy}$  changing sign, and when  $\beta = 0$ , it reduces to Rayleigh's inflection point theorem. (Of course, it may seem fraudulent to use quasi-geostrophic equations to derive Rayleigh's theorem – but actually it's okay. Why?)

The extension of  $\bar{q}$  in (13.46) is reasonable in view of the equivalence of the following situations:

- (i) letting  $\frac{\partial \bar{u}}{\partial z}$  at  $z = 0$  be equal to  $\frac{\partial \bar{u}}{\partial z}$  at  $z = 0+$ , and
- (ii) letting  $\frac{\partial \bar{u}}{\partial z} = 0$  at  $z = 0$  and having a  $\delta$ -function contribution to  $\frac{\partial^2 \bar{u}}{\partial z^2}$  bring  $\frac{\partial \bar{u}}{\partial z}$  at  $z = 0+$  to  $\frac{\partial \bar{u}}{\partial z} = 0$  at  $z = 0$ .

In the second case,  $\tilde{q}$  as defined by (13.46) is actually  $\bar{q}$ , the basic pseudo-potential vorticity. In many cases of practical interest,  $\bar{u}_z > 0$  and the curvature terms in (13.35) are relatively small above the ground, so that  $\bar{q}_y > 0$  in the bulk of the atmosphere. However, the  $\delta$ -function contribution to  $\frac{\partial^2 \bar{u}}{\partial z^2}$  makes  $\bar{q}_y < 0$  at  $z = 0+$ . Thus the surface at which  $\bar{q}_y$  changes sign is just at the ground.

The condition we have derived is only a necessary condition for instability, but, in practice, when it is satisfied we generally do find instability. However, there are important exceptions.

## 13.4 The Kelvin-Orr mechanism

You may have already noticed a certain difference between the convective instability problem we dealt with and the two following sections dealing with problems in plane-parallel shear flow instability: namely, the underlying physics of convective instability was clear (heavy fluid on top of light

fluid), while the physics underlying shear instability was obscure; it is not in the least clear, for example, why changes of potential vorticity gradient should lead to instability. This situation is only beginning to be rectified.

Fortunately, there exists an important example of a shear amplified disturbance for which the physics is relatively clear; interestingly, the disturbance is not a normal mode. We will consider a very simple situation: namely, a basic state consisting in plane parallel flow,  $U(y)$ , in an unbounded, incompressible, non-rotating fluid. In this case we have a stream function,  $\psi$ , for the velocity perturbations, where

$$u = -\frac{\partial\psi}{\partial y} \quad (13.49)$$

$$v = \frac{\partial\psi}{\partial x}. \quad (13.50)$$

Vorticity, given by

$$\begin{aligned} \xi &= \nabla \times \vec{v} \\ &= \nabla^2 \psi \end{aligned} \quad (13.51)$$

is conserved, so

$$\frac{\partial\xi}{\partial t} + U(y)\frac{\partial\xi}{\partial x} = 0. \quad (13.52)$$

For any initial perturbation,

$$\xi(x, y, t = 0) = F(x, y), \quad (13.53)$$

(13.52) will have a solution

$$\xi(x, y, t) = F(x - U(y)t, y); \quad (13.54)$$

that is, the original perturbation vorticity is simply carried by the basic flow. The streamfunction is obtained by inverting (13.51), while the velocity perturbations are obtained from (13.49) and (13.50). The crucial point, thus far, is that it is vorticity (rather than momentum) that is conserved.

For simplicity, we will now take



$$U(y) = Sy, \tag{13.55}$$

and

$$F(x, y) = A \cos(kx). \tag{13.56}$$

Equation 13.54 becomes

$$\begin{aligned} \xi &= A \cos[k(x - Syt)] \\ &= A \{ \cos(kx) \cos(kSty) \\ &\quad + \sin(kx) \sin(kSty) \}. \end{aligned} \tag{13.57}$$

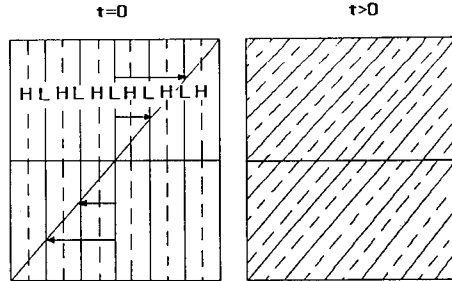


Figure 13.2: A pattern of vertical isolines of vorticity at  $t = 0$  is advected by a constant shear. The dashed lines (H) correspond to high values of vorticity while the solid lines (L) correspond to low values of vorticity. At some  $t > 0$ , the isolines have been rotated in the direction of the shear, and the separation between the isolines has been reduced.

The stream function is easily obtained from (13.57):

$$\psi = \frac{-\xi}{k^2(1 + S^2t^2)}, \tag{13.58}$$

and

$$u = -\frac{\partial\psi}{\partial y} = \frac{ASt}{k(1+S^2t^2)} \sin[k(x-Syt)], \quad (13.59)$$

$$v = \frac{\partial\psi}{\partial x} = \frac{A}{k(1+S^2t^2)} \sin[k(x-Syt)]. \quad (13.60)$$

In general, the above solution represents algebraic decay with time. What is happening is illustrated in Figure 13.2.

Phase lines, which are initially vertical, are rotated by the basic shear, and this causes the lines to move closer together. Now vorticity is conserved and vorticity is made up of derivatives of velocity. So, as the phase lines move closer together, velocity amplitudes must decrease so that the velocity derivatives will still yield the same vorticity. You might plausibly wonder why we are devoting so much attention to a mechanism which leads to decay. But, consider next a slightly more general initial perturbation:

$$F(x, y) = A \cos(kx) \cos(my). \quad (13.61)$$

Equation 13.61 may be rewritten:

$$F(x, y) = \frac{A}{2} (\cos(kx - my) + \cos(kx + my)). \quad (13.62)$$

Now the first term on the right-hand side of (13.62) corresponds to (13.57) begun at some finite positive time, and will therefore decay. However, the second term corresponds to (13.57) begun at some finite negative time (i.e., its phase lines are tilted opposite to the basic shear), and this component of the initial perturbation will grow with advancing time until the phase lines are vertical. After this time, it too will decay; however, for sufficiently large  $m$ , the total perturbation velocity will grow initially (you will calculate how large  $m$  must be as an exercise), and it may grow a great deal before eventual decay sets in.

This mechanism was first discovered by Kelvin (1887); Orr (1907) suggested that this mechanism might explain why flows that are stable with respect to normal mode instabilities still become turbulent. It has been periodically rediscovered since then – but generally dismissed as less effective than exponential growth ‘forever’. However, no disturbance really grows forever, and if the Orr mechanism leads to enough growth to lead to nonlinearity it will be important and the ‘eventual’ decay may be irrelevant. Most recently, Farrell (1987) has been arguing that the Orr mechanism is a major

factor in explosive cyclogenesis, and Lindzen (1988a) has been arguing that the Orr mechanism is the underlying physical mechanism for normal mode instabilities. Unfortunately, the discussion of these tantalizingly important matters is beyond the scope of these notes.

### 13.5 Two-level baroclinic model

One of the simplest models of baroclinic instability is the two-level model for the instability of a basic state  $\bar{u}(z)$  in a Boussinesq fluid (where  $\rho = \text{constant}$ , but  $N^2 \neq 0$ ). For such a fluid, (13.83) and (13.84) become

$$\left(\frac{\partial}{\partial t} + u_G \frac{\partial}{\partial x} + v_G \frac{\partial}{\partial y}\right) \left(\frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} + f\right) - f_0 \frac{\partial w}{\partial z} = 0 \quad (13.63)$$

$$\left(\frac{\partial}{\partial t} + u_G \frac{\partial}{\partial x} + v_G \frac{\partial}{\partial y}\right) \frac{\partial \Phi}{\partial z} + w N^2 = 0. \quad (13.64)$$

We next seek linearized perturbations, about  $u = \bar{u}(z)$ , of the form  $e^{ik(x-ct)}$ :

$$ik(\bar{u} - c) \frac{\partial^2 \Phi'}{\partial x^2} + \beta \frac{\partial \Phi'}{\partial x} = f_0^2 \frac{\partial w'}{\partial z} \quad (13.65)$$

$$ik(\bar{u} - c) \frac{\partial \Phi'}{\partial z} + \underbrace{\frac{1}{f_0} \frac{\partial \Phi'}{\partial x} \frac{\partial}{\partial z} \frac{\partial \bar{\Phi}}{\partial y}}_{\frac{\partial \Phi'}{\partial x} \frac{d\bar{u}}{dz}} + w' N^2 = 0, \quad (13.66)$$

from which we obtain

$$ik\{-k^2(\bar{u} - c) + \beta\} \Phi' = f_0^2 \frac{\partial w'}{\partial z}, \quad (13.67)$$

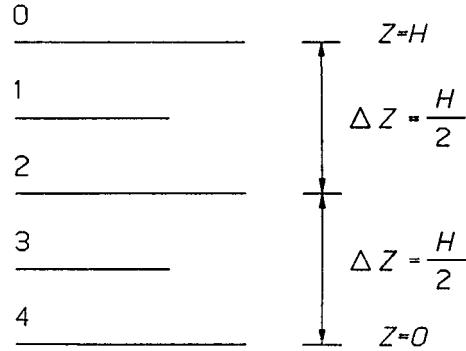
$$ik \left\{ (\bar{u} - c) \frac{\partial \Phi'}{\partial z} - \Phi' \frac{d\bar{u}}{dz} \right\} = -N^2 w'. \quad (13.68)$$

The solution of (13.67) and (13.68), even for  $\bar{u}_z = \text{constant}$ , is quite difficult. However, Phillips (1954) introduced an exceedingly crude difference approximation for which easy solutions can be obtained. Phillips considered a fluid of depth  $H$  where  $w' = 0$  at  $z = 0$  and  $z = H$ . The vertical domain is discretized by five levels (from which we get the name two-level model):

$\bar{u}$  and  $\Phi'$  are evaluated at levels 1 and 3, while  $w'$  is evaluated at level 2 ( $w' = 0$  at levels 0 and 4). Applying (13.67) at level 1 we get

$$ik\{-k^2(\bar{u}_1 - c) + \beta\} \Phi_1 = f_0^2 \frac{w_0 - w_2}{\Delta z} = -f_0^2 \frac{w_2}{\Delta z}. \quad (13.69)$$

Applying (13.68) at level 2 we get



$$ik \left\{ \left( \frac{\bar{u}_1 + \bar{u}_3}{2} - c \right) \frac{\Phi_1 - \Phi_3}{\Delta z} - \left( \frac{\Phi_1 + \Phi_3}{2} \right) \left( \frac{\bar{u}_1 - \bar{u}_3}{\Delta z} \right) \right\} = -N^2 w_2,$$

or

$$ik \{ (\bar{u}_3 - c) \Phi_1 - (\bar{u}_1 - c) \Phi_3 \} = -N^2 \Delta z w_2, \quad (13.70)$$

and evaluating (13.67) at level 3 we get

$$ik \{ -k^2 (\bar{u}_3 - c) + \beta \} \Phi_3 = f_0^2 \frac{w_2 - w_4}{\Delta z} = f_0^2 \frac{w_2}{\Delta z}. \quad (13.71)$$

We may now use (13.69) and (13.71) to reduce (13.70) to a single equation in  $w_2$ :

$$ik \left\{ (\bar{u}_3 - c) \frac{-f_0^2 w_2 / \Delta z}{ik \{ -k^2 (\bar{u}_1 - c) + \beta \}} - (\bar{u}_1 - c) \frac{f_0^2 w_2 / \Delta z}{ik \{ -k^2 (\bar{u}_3 - c) + \beta \}} \right\} = -N^2 \Delta z w_2,$$

or

$$\{ (\bar{u}_3 - c) (-k^2 (\bar{u}_3 - c) + \beta) + (\bar{u}_1 - c) (-k^2 (\bar{u}_1 - c) + \beta) \}$$

$$= \lambda^2(-k^2(\bar{u}_1 - c) + \beta)(-k^2(\bar{u}_3 - c) + \beta), \quad (13.72)$$

where

$$\lambda^2 \equiv N^2 \frac{\Delta^2 z}{f_0^2}.$$

(It should be a matter of some concern that the fundamental horizontal scale length in this problem,  $\lambda$ , depends on the vertical interval,  $\Delta z$ , which is simply a property of our numerical procedure.) Equation 13.72 is a quadratic equation in  $c$ , which after some manipulation, may be rewritten

$$\begin{aligned} c^2\{2k^2 + \lambda^2 k^4\} &+ c\{-(4k^2 + 2\lambda^2 k^4)u_M + 2\beta(1 + k^2 \lambda^2)\} \\ &+ \{u_M^2(2k^2 + \lambda^2 k^4) + u_T^2(2k^2 - \lambda^2 k^4) \\ &- 2u_M\beta(1 + k^2 \lambda^2) + \beta^2 \lambda^2\} = 0, \end{aligned} \quad (13.73)$$

where

$$u_M = \frac{\bar{u}_1 + \bar{u}_3}{2} \quad (13.74)$$

$$u_T = \frac{\bar{u}_1 - \bar{u}_3}{2}. \quad (13.75)$$

Solving (13.73) for  $c$  we get

$$\begin{aligned} c &= u_M - \frac{\beta(1 + k^2 \lambda^2)}{(2k^2 + \lambda^2 k^4)} \\ &\pm \underbrace{\left\{ \frac{\beta^2}{k^4(2 + \lambda^2 k^2)^2} - u_T^2 \frac{(2 - \lambda^2 k^2)}{(2 + \lambda^2 k^2)} \right\}^{1/2}}_{\equiv \delta^{1/2}}. \end{aligned} \quad (13.76)$$

We will have instability when  $\delta < 0$ ; that is, when

$$u_T^2 > \frac{\beta^2}{k^4(4 - \lambda^4 k^4)}. \quad (13.77)$$

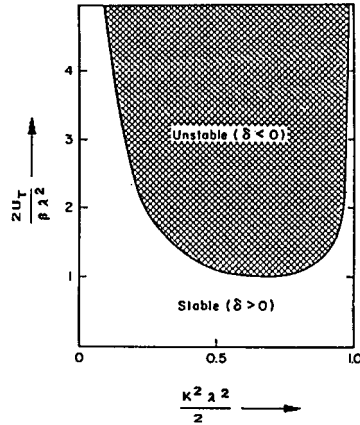


Figure 13.3: Neutral stability curve for the two-level baroclinic model.

The instability diagram for this problem is shown in Figure 13.3. The minimum value of  $u_T^2$  needed for instability is

$$u_{T \min}^2 = \frac{\beta^2 \lambda^4}{4}. \tag{13.78}$$

Equation 13.78 suggests that a minimum shear is needed for instability whereas Subsection 13.3.1 suggested that instability could exist for any finite shear, however small. Actually (13.78) is consistent with our earlier result. Note that for (13.63) and (13.64),

$$\bar{q}_y = \beta - \frac{f_0^2}{N^2} \frac{d^2 \bar{u}}{dz^2}. \tag{13.79}$$

Now, until recently, it was assumed that the basic flow in the ‘two-level’ model corresponded to a constant shear flow characterized by the estimated shear at level 2:  $\frac{\bar{u}_1 - \bar{u}_3}{\Delta z}$ . However, a closer study of the ‘two-level’ model shows that its basic flow has this shear only at level 2; at levels 0 and 4 the relevant basic shear is zero. Thus if we evaluate (13.79) over the upper layer, we get

$$\begin{aligned}\bar{q}_y &= \beta - \frac{f_0^2}{N^2} \left( \frac{0 - \frac{2u_T}{\Delta z}}{\Delta z} \right) \\ &= \beta + \frac{2f_0^2}{N^2} \frac{u_T}{(\Delta z)^2} > 0 \quad \text{for } u_T > 0.\end{aligned}$$

In the lower layer

$$\begin{aligned}\bar{q}_y &= \beta - \frac{f_0^2}{N^2} \left( \frac{\frac{2u_T}{\Delta z} - 0}{\Delta z} \right) \\ &= \beta - \frac{2f_0^2 U_T}{N^2 (\Delta z)^2}.\end{aligned}$$

In order for  $\bar{q}_y$  to change sign we must have

$$u_T > \frac{\beta N^2 (\Delta z)^2}{2f_0^2} = \frac{\beta \lambda^2}{2},$$

or

$$u_T^2 > \frac{\beta^2 \lambda^4}{4},$$

which is precisely what (13.78) says. The difference between the two-level model and a continuous fluid stems from the fact that  $\bar{q}_y$  must be negative over a layer of thickness  $\Delta z$  in order to obtain instability in the two-level model; in a continuous fluid it suffices for  $\bar{q}_y$  to be negative over an arbitrarily thin layer near the surface.

## 13.6 Baroclinic instability and climate

It was already suggested by Phillips that the atmosphere might be trying to achieve baroclinic neutrality, and that this would determine the meridional temperature distribution. In terms of a two-level model one gets



$$\begin{aligned}
 \frac{\partial \bar{T}}{\partial y} &= -\frac{f_0 T_0}{g} \frac{\partial \bar{u}}{\partial z} = -\frac{f_0 T_0}{g} \frac{\beta \lambda^2}{\Delta z} \\
 &= -\frac{f_0 T_0}{g} \frac{\beta}{\Delta z} \frac{(\Delta z)^2 N^2}{f_0^2} = -\frac{f_0 T_0}{g} \frac{\beta}{\Delta z} \frac{(\Delta z)^2}{f_0^2} \left( \frac{\partial \bar{T}}{\partial z} + \frac{g}{c_p} \right) \\
 &= \frac{\beta \Delta z}{f_0} \left( \frac{\partial \bar{T}}{\partial z} + \frac{g}{c_p} \right). \tag{13.80}
 \end{aligned}$$

If we take (13.80) to be locally true at each latitude then

$$\frac{1}{a} \frac{\partial \bar{T}}{\partial \phi} = -\frac{2\Omega \cos \phi \Delta z}{2\Omega \sin \phi} \left( \frac{\partial \bar{T}}{\partial z} + \frac{g}{c_p} \right)$$

or (with obvious cancellation)

$$\frac{\partial \bar{T}}{\partial \phi} = -\frac{\cos \phi \Delta z}{\sin \phi} \left( \frac{\partial \bar{T}}{\partial z} + \frac{g}{c_p} \right). \tag{13.81}$$

If we take  $\Delta z \approx 5$  km and  $\frac{\partial \bar{T}}{\partial z} \approx -6.5^\circ/\text{km}$ , then (13.81) fairly uniformly underestimates the observed  $\frac{\partial \bar{T}}{\partial \phi}$  at the surface by about a factor of 2 for  $\phi \gtrsim 20^\circ$ . This discrepancy becomes worse when one recalls that (13.81) applies to level 2 and that the average of  $\frac{\partial \bar{T}}{\partial \phi}$  over the whole domain will be less than (13.81). On the other hand, (13.81) overestimates  $\frac{\partial \bar{T}}{\partial \phi}$  as one approaches the equator. (Indeed (13.81) blows up at  $\phi = 0$ .)

Despite these problems, there are a number of reasons not to become discouraged with the suggestion that baroclinic neutrality may be relevant to climate:

- (i) Recent numerical experiments with a nonlinear two-level model strongly suggest that when  $\frac{\partial \bar{T}}{\partial \phi}$  for radiative equilibrium exceeds  $\frac{\partial \bar{T}}{\partial \phi}$  for baroclinic neutrality the system approaches baroclinic neutrality.
- (ii) In the tropics we expect  $\frac{\partial \bar{T}}{\partial \phi}$  to be determined by the Hadley circulation – not by baroclinic instability (*viz.* (7.35)).
- (iii) *A priori* we do not expect the two-level model to be quantitatively appropriate to a continuous atmosphere.

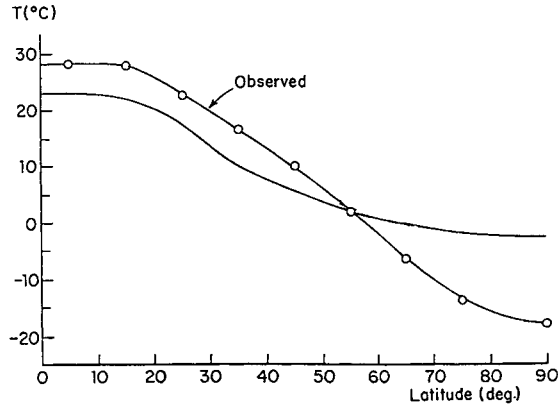


Figure 13.4: Temperature *vs.* latitude. Both the observed distribution and the result of a Hadley-baroclinic adjustment assuming constant  $N^2$  are shown.

A study by Lindzen and Farrell (1980) led to the following conclusion (roughly stated): For a continuous atmosphere with radiative forcing confined below some tropopause at  $z = z_B$ , the appropriate baroclinically neutral profile is one where

$$\bar{q}_y = \beta - e^{z/H} \frac{\partial}{\partial z} \left( \frac{f_0^2}{N^2} e^{-z/H} \frac{\partial \bar{u}}{\partial z} \right) = 0 \text{ below } z_B \quad (13.82)$$

and where

$$\bar{u}_z = 0 \text{ at } z = 0$$

(*viz.* (13.78)). The solution of (13.82) leads to a distribution of  $\frac{\partial \bar{u}}{\partial z}$  at each latitude which in turn leads to a distribution of  $\frac{\partial T}{\partial \phi}$  via the thermal wind relation. The *baroclinically adjusted*  $\left(\frac{\partial T}{\partial \phi}\right)_B$  is taken to be the density weighted average of the distribution of  $\frac{\partial T}{\partial \phi}$  between  $z = 0$  and  $z = z_B$ . We may anticipate that  $\left(\frac{\partial T}{\partial \phi}\right)_B$  will come close to observations for  $\phi \gtrsim 20^\circ$ . Crudely stated, Lindzen and Farrell then took the Hadley-baroclinically adjusted  $\left(\frac{\partial T}{\partial \phi}\right)_{B-H}$

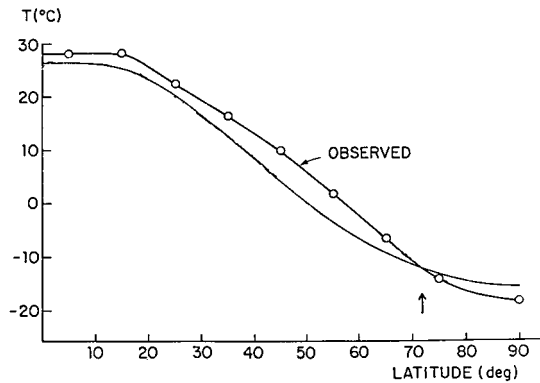


Figure 13.5: Temperature v. latitude. Both the observed distribution and the result of a Hadley-baroclinic adjustment allowing for enhanced  $N^2$  near the surface at high latitudes are shown.

to be the smaller of  $\left(\frac{\partial T}{\partial \phi}\right)_{Hadley}$  and  $\left(\frac{\partial T}{\partial \phi}\right)_{Baroclinic}$ . The resulting global distribution of  $\frac{\partial T}{\partial \phi}$  can be integrated to yield  $T(\phi)$  within an integration constant. This constant is chosen so that outgoing radiation integrated over the globe equals incoming radiation integrated over the globe. Results are shown in Figure 13.4; the Hadley-baroclinic temperature distribution still implies too great a heat flux. Lindzen and Farrell then noted that the assumption that  $N^2$  was constant was not correct. Within a few kilometers of the surface, the atmosphere is substantially more stable than average at high latitudes – especially over ice-covered surfaces. This feature is readily incorporated<sup>1</sup>. Doing this, Lindzen and Farrell obtained the adjusted temperatures shown in Figure 13.5. The agreement is remarkably good and strongly suggests that the observed  $T(\phi)$  is largely determined by Hadley convection and baroclinic neutralization. This result is somewhat surprising insofar as current observations suggest that oceanic currents, stationary waves and transient eddies all contribute comparably to the equator–pole heat flux. However, our results

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<sup>1</sup>Of course one would like a model to predict stability – but the present approach has at least a measure of consistency.

suggest that baroclinic instability acts as a temperature regulator for the whole system – contributing only what is needed, in addition to the contributions of other processes to the heat flux, to achieve ‘baroclinic neutrality’. This suggests that the climate is largely determined by processes which mix potential vorticity – among which processes baroclinic instability is a major contributor. These ideas are still in a rather preliminary and controversial stage, but support for them exists in numerical experiments with general circulation models (Manabe and Strickler, 1965; Manabe and Terpstra, 1974; and others) which show that the total equator–pole heat flux is not particularly sensitive to the inclusion of mountains, hydrology, and so forth – though the makeup of the heat flux, naturally enough, is. At the heart of the above problem is the nonlinear evolution of the instabilities we have touched on in this chapter.

## 13.7 Geometric stabilization

Before ending this chapter, it may be worth noting that the condition for baroclinic neutrality is not the only possible condition. Both Equation 13.77 and Figure 13.3 suggest an alternative approach to stabilization. Figure 13.3 clearly shows the existence of a short-wave cutoff for baroclinic instability in a 2-level model. Instability disappears if

$$\frac{k^2 \lambda^2}{2} > 1.$$

Now, if our fluid is confined within a channel of width  $L$ , then we will have a meridional wavenumber,  $\ell$ , in addition to the zonal wavenumber,  $k$ . Thus,  $k$  in the above equation, must be replaced by the total wavenumber,  $K = (k^2 + \ell^2)^{1/2}$ , and the above condition becomes

$$\frac{K^2 \lambda^2}{2} > 1.$$

Moreover,  $\ell \geq \pi/L$ , and, hence,

$$\frac{K^2 \lambda^2}{2} > \frac{(\pi/L)^2 \lambda^2}{2} > 1.$$

Thus one might *geometrically* stabilize the fluid by confining it in a sufficiently narrow channel. Similarly, for a fixed  $L$ , one could stabilize the fluid by raising the upper boundary and thereby increasing  $\lambda$ .

The situation is not quite so simple as the above suggests. If  $\beta \neq 0$ , then the continuous problem (as opposed to the 2-level problem) does not have a short-wave cutoff. The continuous problem is known as the Charney Problem (Charney, 1947). However, a special case of the continuous problem wherein  $\beta = 0$  or, more generally,  $\bar{q}_y = 0$ , and where the fluid has an upper boundary at a finite height (known as the Eady Problem (Eady, 1947)) does have a short-wave cutoff. Lorenz (1962), noting the relevance of the Eady Problem to baroclinic instability in a rotating annulus, showed that reducing the width of the annulus could stabilize the waves. In the Eady Problem, instability arises from the delta-function contributions to  $\bar{q}_y$  at the top and bottom boundaries.

As it turns out, baroclinic waves in the atmosphere are meridionally confined, not only by the finite extent of the earth, but by the jet like structure of the mean zonal wind (Ioannou and Lindzen, 1986, 1990). Lindzen (1992) has suggested that the atmosphere could be stabilized with respect to baroclinic instability, while maintaining surface temperature gradients, by eliminating  $\bar{q}_y$  in the bulk of the troposphere while concentrating  $\bar{q}_y$  at some upper surface whose height is sufficiently great. This height turns out to be of the order of the tropopause. Observations (Hoskins, et al, 1985) do indeed suggest that at midlatitudes  $\bar{q}_y$  is much smaller in the bulk of the troposphere than it is at either the surface or at the tropopause though  $\bar{q}_y$  in most of the troposphere is still on the order of  $\beta$ . The implications of *geometric* stabilization are still being explored.

## 13.8 Energetics of meteorological disturbances

Before concluding these notes, it probably behooves us to consider the energetics of meteorological disturbances. Discussions of energetics have been a standard component of dynamic meteorology for about 50 years. A consideration of energetics does offer some insights into dynamics, but it has contributed little to the actual solution of problems, and introduces a certain amount of confusion for reasons we will discuss.

Consistent with the approximations of Section 13.3 we may write the quasi-geostrophic equations of motions as follows:

$$\left(\frac{\partial}{\partial t} + \vec{v}_G \cdot \nabla\right) (\zeta_G + f) - f_0 e^{z/H} \frac{\partial}{\partial z} (e^{-z/H} w) = 0 \quad (13.83)$$

$$\left(\frac{\partial}{\partial t} + \vec{v}_G \cdot \nabla\right) \Phi_z + N^2 w = 0. \quad (13.84)$$

Presumably, these equations have an energy integral. To find this integral, let's rewrite (13.83) and (13.84):

$$\nabla^2 \psi_t + \nabla \cdot (\vec{v}_G (\nabla^2 \psi + f)) - f_0 e^{z/H} \frac{\partial}{\partial z} (e^{-z/H} w) = 0 \quad (13.85)$$

$$\psi_{zt} + \nabla \cdot (\vec{v}_G \psi_z) + \frac{N^2}{f_0} w = 0, \quad (13.86)$$

where

$$\psi \equiv \frac{\Phi}{f_0}.$$

(Recall,  $\nabla \cdot \vec{v}_G = 0$ .) Now multiply (13.85) by  $e^{-z/H} \psi$ :

$$\begin{aligned} e^{-z/H} \psi \nabla^2 \psi_t + e^{-z/H} \psi \nabla \cdot (\vec{v}_G (\nabla^2 \psi + f)) \\ - f_0 \psi \frac{\partial}{\partial z} (e^{-z/H} w) = 0. \end{aligned} \quad (13.87)$$

Now

$$\begin{aligned} \psi \nabla^2 \psi_t &= \nabla \cdot (\psi \nabla \psi_t) - \nabla \psi \cdot \nabla \psi_t \\ &= \nabla \cdot (\psi \nabla \psi_t) - \frac{1}{2} \frac{\partial}{\partial t} (|\nabla \psi|^2) \end{aligned}$$

and

$$\begin{aligned} \psi \nabla \cdot (\vec{v}_G (\nabla^2 \psi + f)) &= \nabla \cdot (\psi \vec{v}_G (\nabla^2 \psi + f)) \\ &\quad - (\nabla^2 \psi + f) \underbrace{\vec{v}_G \cdot \nabla \psi}_{=0}. \end{aligned}$$

So (13.87) becomes

$$\begin{aligned}
 \frac{\partial}{\partial t} \left( e^{-z/H} \frac{|\nabla\psi|^2}{2} \right) &= e^{-z/H} \nabla \cdot (\psi \nabla \psi_t) \\
 &\quad + e^{-z/H} \nabla \cdot (\psi \vec{v}_G (\nabla^2 \psi + f)) \\
 &\quad - f_0 \psi \frac{\partial}{\partial z} (e^{-z/H} w). \tag{13.88}
 \end{aligned}$$

Now integrate (13.88) over  $y_1, y_2$ , over  $z$  between 0 and  $\infty$ , and over  $x$  (re-calling periodicity in  $x$ ):

$$\begin{aligned}
 &\frac{\partial}{\partial t} \int \int \int e^{-z/H} \frac{1}{2} |\nabla\psi|^2 dx dy dz \\
 = &\underbrace{\int \int \int e^{-z/H} \nabla \cdot (\psi \nabla \psi_t) dx dy dz}_{\rightarrow 0 \text{ because } \int_x \int_z e^{-z/H} \frac{\partial u_G}{\partial t} dx dz|_{y=y_1, y_2=0}} \\
 &+ \underbrace{\int \int \int e^{-z/H} \nabla \cdot (\psi \vec{v}_G (\nabla^2 \psi + f)) dx dy dz}_{\rightarrow 0 \text{ because normal velocities at boundaries } = 0} \\
 &- f_0 \int \int \int \psi \frac{\partial}{\partial z} (e^{-z/H} w) dx dy dz \tag{13.89}
 \end{aligned}$$

Clearly, the left hand side of (13.89) is the time rate of change of kinetic energy. We turn next to (13.86). Multiply (13.86) by  $e^{-z/H} \frac{f_0}{N^2} \psi_z$ :

$$\begin{aligned}
 e^{-z/H} \frac{f_0^2}{N^2} \psi_z \psi_{zt} &+ e^{-z/H} \frac{f_0^2}{N^2} \psi_z \nabla \cdot (\vec{v}_G \psi_z) \\
 &+ f_0 e^{-z/H} \psi_z w = 0. \tag{13.90}
 \end{aligned}$$

Now  $\psi_z \psi_{zt} = \frac{1}{2} \frac{\partial}{\partial t} (|\psi_z|^2)$  and

$$\psi_z \nabla \cdot (\vec{v}_G \psi_z) = \frac{1}{2} \nabla \cdot (\vec{v}_G |\psi_z|^2).$$

So (13.90) becomes

$$\begin{aligned} & \frac{\partial}{\partial t} \left( e^{-z/H} \frac{f_0^2}{N^2} \frac{1}{2} |\psi_z|^2 \right) \\ &= -e^{-z/H} \frac{f_0^2}{N^2} \frac{1}{2} \nabla \cdot (\vec{v}_G |\psi_z|^2) - f_0 e^{-z/H} \psi_z w. \end{aligned} \quad (13.91)$$

Integrating (13.91) over  $x, y, z$  we get

$$\begin{aligned} & \frac{\partial}{\partial t} \int \int \int e^{-z/H} \frac{f_0^2}{N^2} \frac{1}{2} |\psi_z|^2 dx dy dz \\ &= - \underbrace{\int \int \int e^{-z/h} \frac{f_0^2}{N^2} \frac{1}{2} \nabla \cdot (\vec{v}_G |\psi_z|^2) dx dy dz}_{=0 \text{ (Why?)}} \\ & \quad - f_0 \int \int \int \psi_z w e^{-z/H} dx dy dz. \end{aligned} \quad (13.92)$$

The left hand side of (13.92) is the time rate of change of something called the *available potential energy*. We will explain what this is in the next section.

Finally, adding (13.89) and (13.92) we get

$$\begin{aligned} & \frac{\partial}{\partial t} \int \int \int e^{-z/H} \left\{ \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} \frac{f_0^2}{N^2} |\psi_z|^2 \right\} dx dy dz \\ &= -f_0 \underbrace{\int \int \int \frac{\partial}{\partial z} (\psi w e^{-z/H}) dx dy dz}_{=0 \text{ (Why?)}}. \end{aligned} \quad (13.93)$$

The quantity

$$E = \frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} \frac{f_0^2}{N^2} |\psi_z|^2 \quad (13.94)$$



is the total energy per unit mass.

Energy budgets for the atmosphere and oceans are studied at great length, but the interpretation of such studies must be approached cautiously. For example, if we divide our fields into zonally averaged parts and eddies,

$$\psi = \bar{\psi}(y, z, t) + \psi'(x, y, z, t)$$

$$T = \bar{T} + T'$$

$$w = \bar{w} + w'$$

$$\zeta = \bar{\zeta} + \zeta',$$

where  $(\bar{\quad}) \equiv$  zonal average of  $(\quad)$ , we can obtain, after modest amounts of algebra<sup>2</sup>,

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{y_1}^{y_2} \int_0^\infty e^{-z/H} \overline{\left\{ \frac{1}{2}(\nabla\psi')^2 + \frac{1}{2} \frac{f_0^2}{N^2} \left( \frac{\partial\psi'}{\partial z} \right)^2 \right\}} dydz \\ &= \int_{y_1}^{y_2} \int_0^\infty e^{-z/H} \overline{\psi'_x \psi'_y} \bar{u}_y dydz - \int_{y_1}^{y_2} \int_0^\infty \frac{f_0^2}{N^2} \overline{\psi'_x \psi'_z} \bar{\psi}_{zy} e^{-z/H} dydz \\ &= - \int_{y_1}^{y_2} \int_0^\infty e^{-z/H} \overline{u'v'} \bar{u}_y dydz \\ & \quad - \int_{y_1}^{y_2} \int_0^\infty \frac{g/T_0}{(\langle \frac{\partial T}{\partial z} \rangle + \frac{g}{c_p})} \overline{v'T'} \bar{T}_y e^{-z/H} dydz, \end{aligned} \tag{13.95}$$

where  $T_0 =$  average of  $T$  over whole fluid, and  $\langle T \rangle =$  horizontal average of  $T$ .

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<sup>2</sup>The procedure consists in averaging Equations 13.83 and 13.84 with respect to  $x$ , subtracting these averages from (13.83) and (13.84), multiplying the resulting equations by  $-e^{-z/H}\psi'$  and  $(f_0^2/N^2)e^{-z/H}\psi'_z$ , respectively, adding the two resulting equations, and integrating over the fluid volume. The reader should go through the derivation of both Equations 13.95 and 13.96.

It can also be shown that

$$\begin{aligned}
 & \frac{\partial}{\partial t} \int_{y_1}^{y_2} \int_0^\infty e^{-z/H} \left\{ \frac{1}{2} (\nabla \bar{\psi})^2 + \frac{1}{2} \frac{f_0^2}{N^2} (\bar{\psi}_z)^2 \right\} dy dz \\
 &= \int_{y_1}^{y_2} \int_0^\infty e^{-z/H} \overline{u'v'} \bar{u}_y dy dz \\
 &+ \int_{y_1}^{y_2} \int_0^\infty \frac{g/T_0}{(\langle \frac{\partial T}{\partial z} \rangle + \frac{g}{c_p})} \overline{v'T'} \bar{T}_y e^{-z/H} dy dz. \quad (13.96)
 \end{aligned}$$

Equations 13.95 and 13.96, together, seem to tell us that the sum of the energy in the zonally average flow and the eddies is conserved, and that the growth of eddies occurs at the expense of the mean flow through the action of horizontal eddy stresses on  $\bar{u}_y$  and horizontal eddy heat fluxes on  $\bar{T}_y$ . Although these remarks are subject to interpretation (and we will discuss them later) they do establish the criteria for eddy growth to be energetically consistent; in particular they state that the energy of the zonally averaged flow can only be tapped if  $\bar{u}_y \neq 0$  and/or  $\bar{T}_y \neq 0$  (in the above quasi-Boussinesq system).

It is, however, sometimes concluded that instabilities are caused by  $\bar{u}_y$  and/or  $\bar{T}_y$ . This is not, in general, true. To be sure, if  $\bar{u}_y = \bar{T}_y = 0$  there can be no unstable eddies. In such a case, the necessary condition for instability given in Subsection 13.3.1 would not be satisfied. However, this condition can also fail to be satisfied when  $\bar{u}_y \neq 0$  and/or  $\bar{T}_y \neq 0$ .

Equation 13.95 does tell us that a growing eddy will act to reduce  $\bar{T}_y$  and  $\bar{u}_y$ . The first term on the right hand side of (13.95) is called a barotropic conversion while the second term is called a baroclinic conversion. Instabilities of flows where  $\bar{u}_y = 0$  are called *baroclinic instabilities*, while instabilities where  $\bar{T}_y = 0$  (and  $\bar{u}_z = 0$ ) are called *barotropic instabilities*. However, as we saw in Subsection 13.3.1 both instabilities depend on the meridional gradient of potential vorticity.

## 13.9 Available potential energy

The quantity available potential energy is at first sight a bit strange. We expect the conservation of kinetic + potential + internal energy. Now the

last two items can be written as

$$P = \int \int \int gz \rho \, dzdxdy \quad (13.97)$$

and

$$I = \int \int \int c_v T \rho \, dzdxdy. \quad (13.98)$$

The former can be rewritten

$$\begin{aligned} P = \int \int \int gz \rho \, dzdxdy &= \int \int \int_0^{p_0} z dp = \int \int \int_0^\infty pdz \\ &= \int \int \int RT \rho dz = \frac{R}{c_v} I, \end{aligned} \quad (13.99)$$

and

$$\begin{aligned} I + P &= \text{total potential energy} = TPE \\ &= \left(1 + \frac{c_v}{R}\right) \int \int \int RT \rho \, dzdxdy \\ &= c_p \int \int \int T \rho \, dzdxdy \\ &= \frac{5}{2} \int \int \int c^2 \rho \, dzdxdy, \end{aligned} \quad (13.100)$$

where  $c$  = the sound speed. Clearly  $TPE \gg K$ . Noticing this disparity, Margules (1903) pointed out that without deviations from horizontal stratification *no TPE* is *available* to  $K$  (assuming static *stability*). Lorenz (1955) showed that what we have called available potential energy (*APE*) is that portion of  $TPE$  which is available to  $K$ .

Lorenz's analysis ran roughly as follows (ignoring horizontal integrals)<sup>3</sup>:

$$TPE = \frac{c_p}{g} \int_0^{p_0} T dp = (1 + \kappa)^{-1} \frac{c_p}{g} p_0^{-\kappa} \int_0^\infty p^{1+\kappa} d\Theta,$$

where  $T = \Theta p^\kappa p_0^{-\kappa}$  and  $\Theta$  is the *potential temperature*.

---

<sup>3</sup>We are not particularly concerned with details since we are merely seeking an interpretation of APE.

He then noted that the minimum  $TPE$  which motions would produce would be achieved if  $p = \langle p \rangle$  everywhere ( $\langle \rangle =$  horizontal average); that is,

$$TPE_{min} = (1 + \kappa)^{-1} \frac{c_p}{g} p_0^{-\kappa} \int_0^\infty \langle p \rangle^{1+\kappa} d\Theta.$$

Now let

$$\begin{aligned} \langle APE \rangle &= \langle TPE - TPE_{min} \rangle \\ &= (1 + \kappa)^{-1} \frac{c_p}{g} p_0^{-\kappa} \int_0^\infty (\langle p^{1+\kappa} \rangle - \langle p \rangle^{1+\kappa}) d\Theta \end{aligned}$$

and let

$$\begin{aligned} p &= \langle p \rangle + \tilde{p} \\ p^{1+\kappa} &= \langle p \rangle^{1+\kappa} \left\{ 1 + (1 + \kappa) \frac{\tilde{p}}{\langle p \rangle} + \frac{\kappa(1 + \kappa)}{2!} \frac{\tilde{p}^2}{\langle p \rangle^2} + \dots \right\} \end{aligned}$$

and

$$\langle APE \rangle = \frac{1}{2} \kappa \frac{c_p}{g} p_0^{-\kappa} \int_0^\infty \langle p \rangle^{1+\kappa} \left\langle \frac{\tilde{p}^2}{\langle p \rangle^2} \right\rangle d\Theta \quad (13.101)$$

(i.e.,  $\langle APE \rangle$  depends on the variance of pressure over isentropic surfaces). If one wishes to deal with isobaric rather than isentropic surfaces we may use

$$\tilde{p} \approx \frac{\partial p}{\partial \theta} \tilde{\theta}$$

(where  $\tilde{\theta}$  is the deviation of  $\theta$  from  $\langle \theta \rangle$  on isobaric surfaces) to rewrite (13.101) as

$$\begin{aligned} \langle APE \rangle &\approx \frac{1}{2} \kappa \frac{c_p}{g} p_0^{-\kappa} \int_0^{p_0} \langle p \rangle^{-(1-\kappa)} \langle \theta \rangle^2 \left\{ -\frac{\partial \langle \theta \rangle}{\partial p} \right\}^{-1} \left\langle \frac{\tilde{\theta}^2}{\langle \theta \rangle^2} \right\rangle dp. \end{aligned}$$

Now

$$\frac{\partial \theta}{\partial p} = -\kappa \frac{c_p}{g} \left( \frac{\partial T}{\partial z} + \frac{g}{c_p} \right).$$

Also

$$\frac{\tilde{\theta}}{\bar{\theta}} \approx \frac{\tilde{T}}{\bar{T}}$$

on isobaric surfaces. So

$$\begin{aligned} \langle APE \rangle &= \frac{1}{2} \int_0^{p_0} \langle T \rangle \frac{1}{\left( \langle \frac{\partial T}{\partial z} \rangle + \frac{g}{c_p} \right)} \left\langle \frac{\tilde{T}^2}{\langle T \rangle^2} \right\rangle dp \\ &= \frac{1}{2} \int_0^\infty \rho \frac{g}{\langle T \rangle} \frac{1}{\left( \langle \frac{\partial T}{\partial z} \rangle + \frac{g}{c_p} \right)} \langle \tilde{T}^2 \rangle dz, \end{aligned}$$

which is what we found in Section 13.8.

## 13.10 Some things about energy to think about

For many people there is something comforting about energetics. Yet as we have seen, it is at best a tool to establish consistency rather than causality. The problems go further than this. For example, energy is not Galilean invariant. Clearly, the kinetic energy depends on the frame of reference in which we are measuring velocity. In addition, the quantity in (13.96) whose time derivative is being taken is not the full eddy contribution to energy. You can easily prove this for yourself. Take (13.94) (which is the total energy), and substitute  $\bar{\psi} + \psi'$  for  $\psi$ . The eddy contributions include cross terms which are linear in  $\psi'$  as well as the quadratic terms in (13.96). To be sure, these linear terms go to zero when averaged over  $x$ , but they are nonetheless part of the eddy energy, and their presence in some problems can lead to eddy energy not being positive-definite. This, in itself, is not as disturbing as it might appear at first sight. For example, when an eddy with a phase speed smaller than the mean flow is absorbed and reduces the mean flow speed, shouldn't we think of the eddy before it was absorbed as having negative energy insofar as its absorption led to the reduction of the mean kinetic energy. It should also be clear that this whole process would depend on the

frame of reference in which speed was measured. What then is the quadratic quantity in (13.96)? Well, it is at least a pretty good measure of the overall magnitude of the eddies.

It is likely that the above is almost certain to be more confusing than edifying for many readers. However, it was merely meant to give you something to think about. In doing so you might actually avoid some confusion later on.