12. Shallow water equations with rotation – Poincaré waves

Considering now motions with L<<R, we can write the equations of motion in Cartesian coordinate:

1. $u_t - fv = \frac{1}{\rho_o} p_x$

2. $v_t + fu = -\frac{1}{\rho_o} p_y$

3. $0 = -\frac{1}{\rho_o} p_z - \frac{g\rho}{\rho_o}$

4. $u_x + v_y + w_z = 0$

$\rho_t + w\rho_{oz} = 0$

In the general case:

Again, the vertical momentum equation 3, and the adiabatic equation 5 can be combined:

$$\frac{1}{\rho_o} p_{zt} + N^2 w = 0$$

Consider first a homogeneous fluid for which $\rho_{total} = \rho_o + \rho = \rho_o$ the hydrostatic equation. Then $p_{oz} = -g\rho_o$ is the total pressure. Integrating from z to the free surface η:

$p|_\eta - p(z) = -g\rho_o(\eta - z)$ or

$p(z) = p_{atm} + g\rho_o (\eta - z) \Rightarrow p(z) = g\rho_o (\eta - z)$ as $p_{atm} = 0$

Then the horizontal pressure gradient $\underline{\nabla} p = g\rho \underline{\nabla}\eta$ is independent of z. The equations of motion can be written:

$u_t - fv = -g\eta_x$ (there is no adiabatic equation as there

$v_t + fu = -g\eta_y$ is no perturbation density)

$u_x + u_y + w_z = 0$

Notice that the right-hand side of the horizontal momentum equations is independent of z: (u,v) are independent of z

Integrate the continuity equation $\int_{-D}^{\eta} dz$

$$\int_{-D}^{\eta}(u_x + v_y)dz + w\,|_{z=\eta} - w\,|_{z=-D} = 0$$

Top and bottom b.c. are (including non-linearities)

$$\text{We have: } w = \frac{d\eta}{dt} \text{ at } z = \eta;\; w = \frac{-dD}{dt} = -\vec{u}_H \bullet \underline{\nabla}\, D \text{ at } z = -D$$

As:

$w_{zz} = 0$

$w_z = a(t)$ $\quad$ $w = a(t)z + b(t)$

at z = -D $\quad$ w = 0 $\quad$ -Da+b = 0 $\quad$ b = aD

$w = a(t)(z+D)$

at $z = 0$ $\quad$ $w = \frac{\partial\eta}{\partial t}$ $\quad$ $Da(t) = \frac{\partial\eta}{\partial t}$

$$w = \frac{1}{D}\frac{\partial\eta}{\partial t}(z + D) \qquad \text{linearized version}$$

In general $\quad$ $w = a(x,y,t)(z+D)$

at $z = o\; a = \frac{d\eta}{dt}\frac{1}{D}$

$$w = \frac{1}{D}\frac{d\eta}{dt}(z + D)$$

$$\frac{d\eta}{dt} = \frac{\partial\eta}{\partial t} + \vec{u}_H \bullet \underline{\nabla}\eta$$

and we obtain

$$(u_x + v_y)(\eta + D) + \frac{d\eta}{dt} + \vec{u}_H \bullet \underline{\nabla} D = 0$$

$$(u_x + v_y)(\eta + D) + \frac{d\eta}{dt} + \vec{u}_H \bullet \underline{\nabla}(\eta + D) = 0$$

or $\eta_t + [u(\eta + D)]_x + [v(\eta + D)]_y$ exactly

Assuming $\eta << D$ i.e. linearizing, we have

$$u_t - fv = -g\eta_x$$

$$v_t + fu = -g\eta_y$$

$$\eta_t + (uD)_x + (VD)_y = 0$$

These are the linear shallow water equations for a homogenous fluid with rotation.

Now return to the equations with stratification and separate variables

$$u = U(x,y,t)F(z)$$

$$v = V(x,y,t)F(z)$$

$$p = P(x,y,t)H(z)$$

$$w = W(x,y,t)G(z)$$

We get

$$(U_t - fV)F = \frac{-1}{\rho_o} P_x H$$

$$(V_t + fU)F = -\frac{1}{\rho_o} P_y H$$

$$N^2 WG = -\frac{1}{\rho_o} P_t H_z$$

$$(U_x + V_y)F + WG_z = 0$$

Choose $W = P_t$ (as we derived in LTE)

$H = g\rho_o F \quad G_z = F/D$

We have

$U_t - fV = -gP_x$ Horizontal structure equations

$V_t + fV = -gP_y$

$P_t + D_n(U_x + V_y) = 0$

Compare with the homogeneous layer equations with D=constant.

They are the same with $P = \eta$. The pressure plays the part of the sea/surface elevation.

The vertical structure equation is again:

$$G_{zz} + \frac{N^2(z)}{gD_n} G = o$$

$$G_z - \frac{1}{D_n} G = 0 \text{ at } z = o$$ The same identical as for LTE with $h_n = D_n$

$$G = o \text{ at } z = -D$$

The hydrostatic approximation we have made assumes $w_t << g$ which is equivalent to assuming

$\omega^2 << N^2$ ->

In a flat-bottom ocean stratification makes possible an infinite sequence of internal replicas of the barotropic, long, shallow water gravity waves. We shall study the latter first.

From now on we shall study the homogeneous one layer problem as it is equivalent to the horizontal structure equations ($P=\eta$) for the fully stratified case. With D = constant

$u_t - fv = -g\eta_x$

$v_t + fu = -g\eta_y$

$\eta_t + D(u_x + v_y) = 0$

Form the vorticity equation $\zeta = v_x - u_y$ cross-differentiating the 2 horizontal momentum equations

$$\frac{\partial\zeta}{\partial t} = -f(u_x + v_y) = \frac{f}{D}\frac{\partial\eta}{\partial t}$$ Statement of conservation of potential vorticity

or $$\frac{\partial}{\partial t}(\xi - \frac{f}{D}\eta) = 0$$ for the linear, homogeneous model with f = constant

$$q = \zeta - \frac{f}{D}\eta$$

relative vorticity   vortix stretching

For periodic motions $\frac{\partial}{\partial t}q = -iwq = 0$ q vanishes

Now we want an equation for η: take the divergence of horizontal momentum equations

$$\frac{\partial}{\partial t}(u_x + v_y) - f\zeta = -g\nabla^2\eta$$

From continuity

$$\frac{\partial^2\eta}{\partial t^2} + D\frac{\partial}{\partial t}(u_x + v_y) = 0 \Rightarrow \frac{\partial}{\partial t}(u_x + v_y) = -\frac{1}{D}\frac{\partial^2\eta}{\partial t^2}$$

$$-\frac{1}{D}\frac{\partial^2\eta}{\partial t^2} - f\zeta = -g\nabla^2\eta$$

From the statement for PV

$$\zeta = q + \frac{f}{D}\eta$$

$$-\frac{1}{D}\frac{\partial^2\eta}{\partial t^2} - f(q + \frac{f}{D}\eta) + g\nabla^2\eta = 0$$

$$\nabla^2\eta - \frac{1}{gD}\frac{\partial^2\eta}{\partial t^2} - \frac{f^2}{gD}\eta = +\frac{f}{g}q$$

Note that as $\frac{\partial q}{\partial t} = 0$; potential vorticity is conserved steadily. So we can separate

$\eta = \eta_{steady} + \eta_{wave} = \eta_s + \eta_w$.

The unsteady part of η_{wave} carries no potential vorticity so

$$\nabla^2 \eta_w - \frac{1}{gD}\frac{\partial^2 \eta_w}{\partial t^2} - \frac{f^2}{gD}\eta_w = 0$$ homogeneous equation

$$\eta_s = \frac{D}{f} q = \frac{D}{f} q_o$$ particular solution

Steady part is in geostrophic balance with (u,v) and reflects the initial distribution of PV as $\frac{\partial q}{\partial t} = 0$; $q = q_o$

The wave equation is:

$$\nabla^2 \eta - \frac{1}{c_o^2}\frac{\partial^2 \eta}{\partial t^2} - \frac{f^2}{c_o^2}\eta = 0$$

When $c_o = \sqrt{gD}$ is the phase speed for long gravity waves.

If there were no rotation $f = 0$ we would get the non-dispersive wave equation (one-dimension)

$$\eta_{xx} - \frac{1}{c_o^2}\eta_{tt} = 0$$ with solution

$\eta = F(x-c_o t) + G(x+c_o t)$; F and G determined by initial condition

Taking a solution of the form:

$$\eta = Ae^{i(kx+ly-\omega t)} \qquad K = \sqrt{k^2+l^2}$$

we obtain $\omega^2 = c_o^2(k^2+l^2) + f^2 \Rightarrow \omega = \pm\sqrt{c_o^2(k^2+l^2)+f^2}$

These are long, shallow water gravity waves modified by rotation, often called Poincaré waves. Visualize the particle motion:

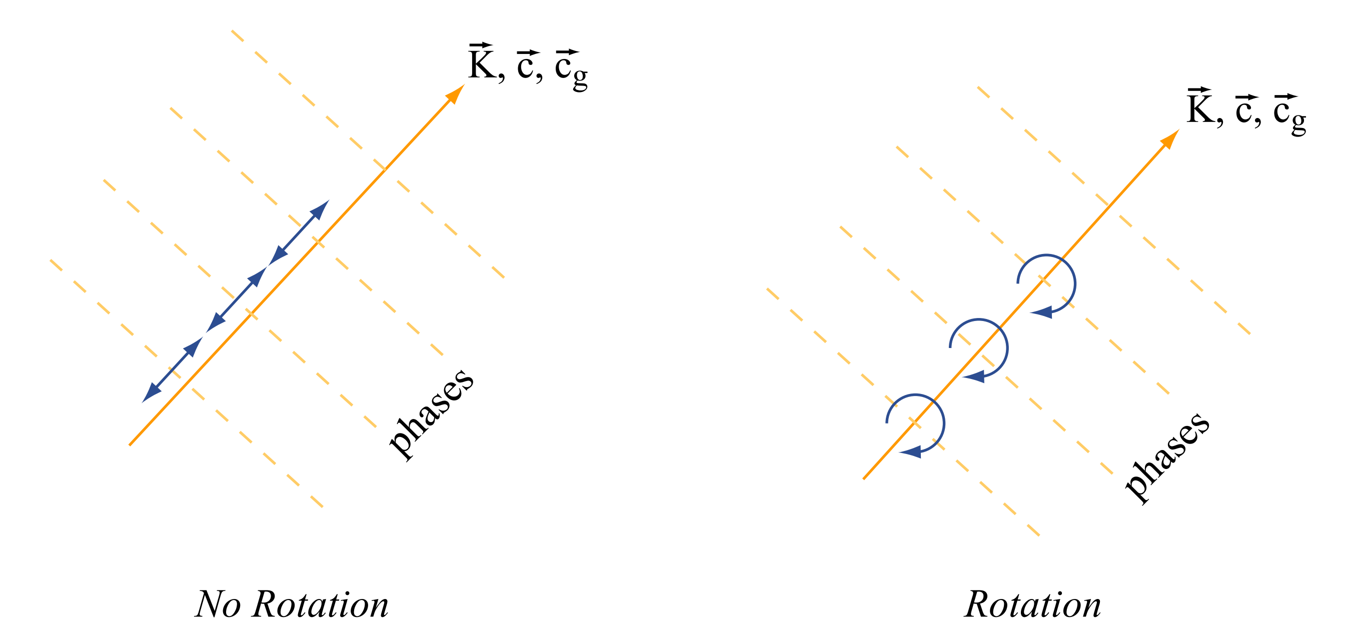

Figure 12-1

Figure by MIT OpenCourseWare.

All these waves have $\omega > f$, <u>f is the lowest possible frequency</u>:

<u>Group velocity</u>

$$c_{gx} = \frac{\partial \omega}{\partial k} = c_o^2 \frac{k}{\omega} = c_o^2 \frac{k}{\sqrt{f^2 + c_o^2 K^2}} \qquad \vec{c}_g // \vec{K}$$

$$c_{gy} = \frac{\partial \omega}{\partial \ell} = c_o^2 \frac{\ell}{\omega} = c_o^2 \frac{\ell}{\sqrt{f^2 + c_o^2 K^2}}$$

The horizontal velocities are obtained

i) eliminating v from momentum eqn. → eq. for u

ii) eliminating u → eq. for v

$$\frac{\partial^2 u}{\partial t^2} + f^2 u = -g\frac{\partial^2 \eta}{\partial x \partial t} - fg\frac{\partial \eta}{\partial y}$$

operator LHS = 0 w = ±f

inertial oscillations

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -g\frac{\partial^2 \eta}{\partial t \partial y} - gf\frac{\partial \eta}{\partial x}$$

u = cos(ft); v = sin(ft)

If we align x with $\vec{K}$ then full solution is

$$\eta = \eta_o \cos(kx - \omega t)$$

$$u = \frac{\eta_o}{D}\frac{\omega}{k}\cos(kx - \omega t)$$

$$v = \frac{\eta_o}{D}\frac{f}{k}\sin(kx - \omega t)$$

Poincaré wave energy → concentrated at lowest possible frequency, near f

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