10. Unbounded domain - non-rotating reflection from a solid boundary

We consider the reflection from a solid boundary which is at some angle with the horizontal. Consider a two-dimensional solution

$e^{-i\omega t+ikx+imz}$ aligning x with the horizontal wave vector k_H

satisfying

$w_{zz} - R^2 w_{xx} = 0.$

with $R^2 = \frac{N^2 - \omega^2}{\omega^2 - f^2}$ and $m = \pm Rk$.

The lines of constant phase are $\theta = kx+mz-\omega t$ = constant or:

$$+kx \pm Rkz - \omega t = \text{constant}$$

that is

$$x \pm Rz = \left(\frac{\omega}{k}\right)t = \text{constant};$$

energy propagates along the lines of constant phase

$x \pm Rz$ = constant that is:

$$z = \frac{1}{R}x \quad \text{(positive slope)} \qquad z = \frac{-1}{R}x \quad \text{(negative slope)}$$

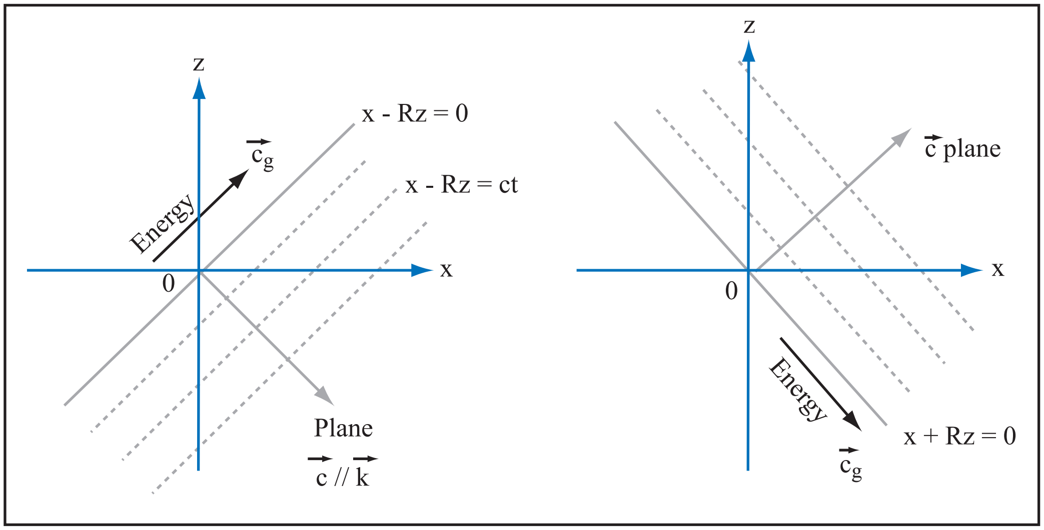
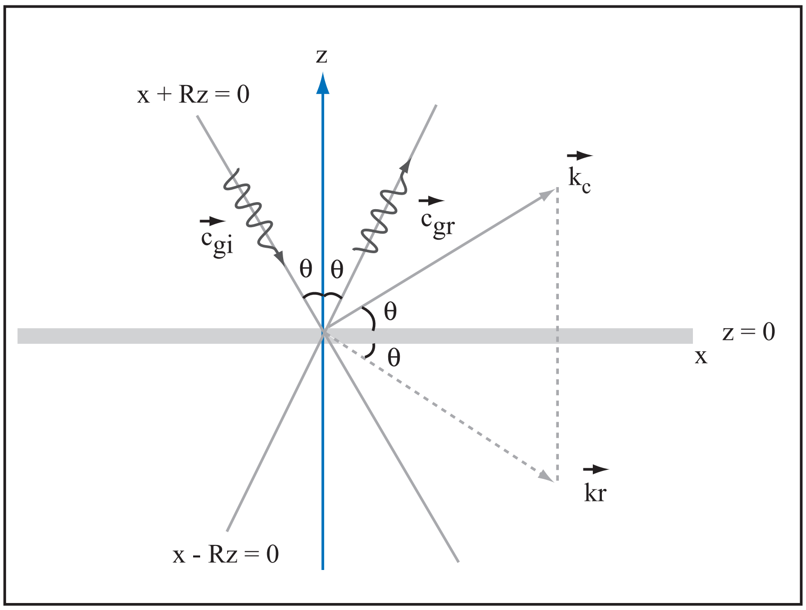


Figure by MIT OpenCourseWare.

These lines are the characteristics of the hyperbolic equation for w, i.e.

$$w = f(x+Rz) + g(x-Rz)$$

Consider first a wave incident and reflected at the horizontal boundary z = 0, i.e. the x-axis



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$\vec{c}_{gi}$ downward: energy propagates along x+Rz=0. The incident wave number $\vec{K}_i$ is perpendicular to $\vec{c}_{gi}$ and upward, Energy is reflected along x-Rz, upward $\vec{c}_{gr}$. The reflected wave number $\vec{K}_r$ is downward.

$\omega = N\cos\theta$ is conserved in the reflection.

θ is the angle of $\vec{K}$ with the horizontal.

As ω is determined only by θ, the angle to the horizontal, $\vec{K}_i$ and $\vec{K}_r$ must form equal angles θ with the horizontal. In this particular case $|\vec{K}_i|\cos\theta = |\vec{K}_r|\cos\theta$.

We can demonstrate that ω is conserved as follows.

Let us consider the more general case of a wall inclined to the horizontal z = ax and let us consider a 2-D problem. Then continuity is simply $u_x+w_z=0$ and we can introduce a streamfunction ψ

$$\begin{cases} u = -\dfrac{\partial\psi}{\partial z} \\ w = +\dfrac{\partial\psi}{\partial x} \end{cases}$$

The incident wave, in terms of ψ, is:

$$\psi_I = \psi_{io} e^{i(k_i x + m_i z - \omega_i t)}$$

and

$$\psi_R = \psi_{ro} e^{i(k_r x + m_r z - \omega_r t)}$$

The total wave field in the reflection is

$$\psi_{Total} = \psi_I + \psi_R$$

and on z = ax ψ_T = constant = 0 without loss of generality. Then

$$\psi_{io}\, e^{i[(k_i + am_i)x - \omega_i t]} +$$

$$+ \psi_{ro} e^{i[(k_r + am_r)x - \omega_r t]} \equiv 0$$

This is true only if

$$\omega_i = \omega_r$$

$k_i + am_i = k_r + am_r$ -> $k_i + \tan\alpha\, m_i = k_r + \tan\alpha\, m_r$

as $a = \tan\alpha$

or $k_i \cos\alpha + m_i \sin\alpha = k_r \cos\alpha + m_r \sin\alpha$

or $\vec{K}_i \bullet \hat{i}_B = \vec{K}_r \bullet \hat{i}_B$ if $\hat{i}_B$ is the unit vector along z = ax

that is:

1. ω is conserved in the reflection process

-> the angle of $\vec{K}_r$ and $\vec{K}_i$ to the horizontal must have the same magnitude θ

2. The component of $\vec{K}_i$ and $\vec{K}_r$ along the slope must be the same

Let us consider the geometry of the process:

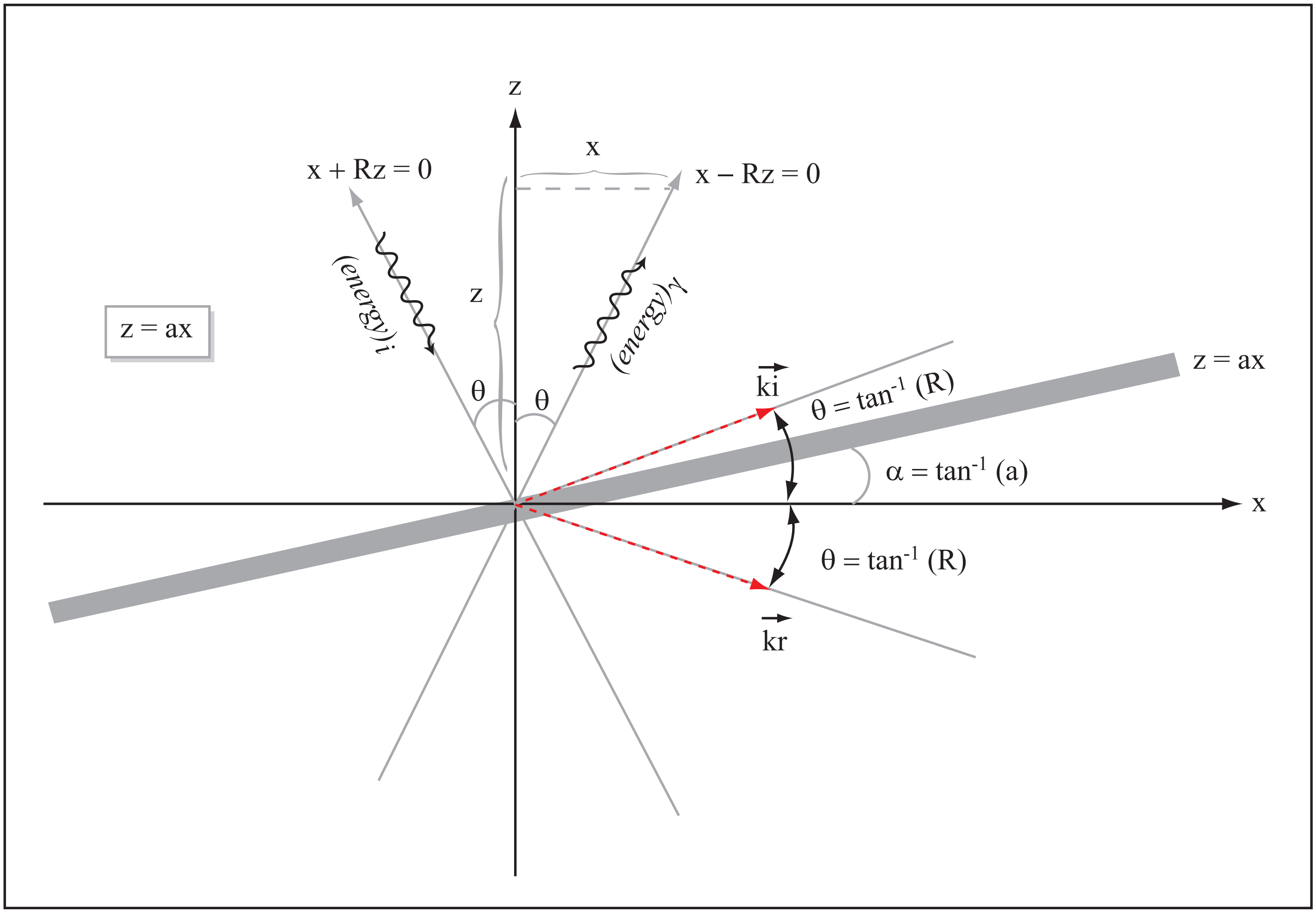


Figure by MIT OpenCourseWare.

$x = z \tan\theta = R z \qquad \tan\theta = R$

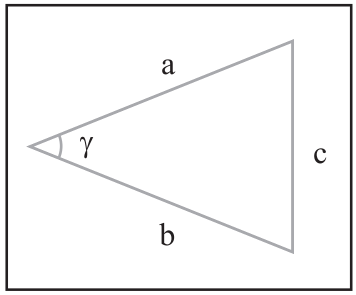
$\theta = \tan^{-1}R \qquad \alpha = \tan^{-1}a$

The projection of $\vec{K}_i, \vec{K}_r$ along the reflecting wall z = ax must be equal:

$$|\vec{K}_i|\cos[\tan^{-1}R - \tan^{-1}a] = |\vec{K}_r|\cos[\tan^{-1}R + \tan^{-1}a]$$

We can evaluate this expression by geometry and the law of cosines:

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}$$



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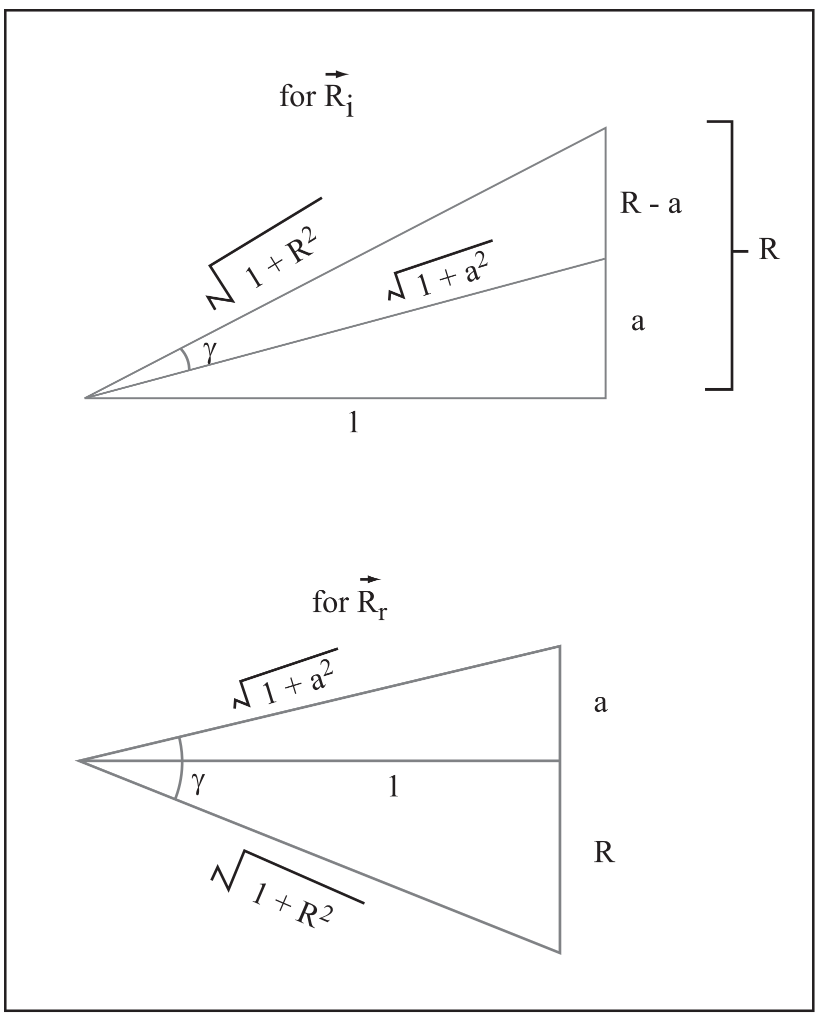
$$\cos(\tan^{-1}R - \tan^{-1}a) = \frac{1 + R^2 + 1 + a^2 - (R - a)^2}{2\sqrt{1 + R^2}\sqrt{1 + a^2}} = \frac{1 + aR}{\sqrt{1 + R^2}\sqrt{1 + a^2}}$$

$$\cos(\tan^{-1}R + \tan^{-1}a) = \frac{1 + a^2 + 1 + R^2 - (R + a)^2}{2\sqrt{1 + R^2}\sqrt{1 + a^2}} = \frac{1 - aR}{\sqrt{1 + R^2}\sqrt{1 + a^2}}$$



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And the above expression becomes:

$$|\vec{K}_i|\frac{1+aR}{\sqrt{1+R^2}\sqrt{1+a^2}} = |\vec{K}_R|\frac{1-aR}{\sqrt{1+R^2}\sqrt{1+a^2}}$$

or

$$|\vec{K}_R| = \frac{1+aR}{1-aR}|\vec{K}_i|$$

or

$$k_R = (\frac{1+aR}{1-aR})k_i$$

$$m_r = -(\frac{1+aR}{1-aR})m_i$$

The reflected wave number $|\vec{K}_R| > |\vec{K}_i|$

=> $\lambda_R < \lambda_i$

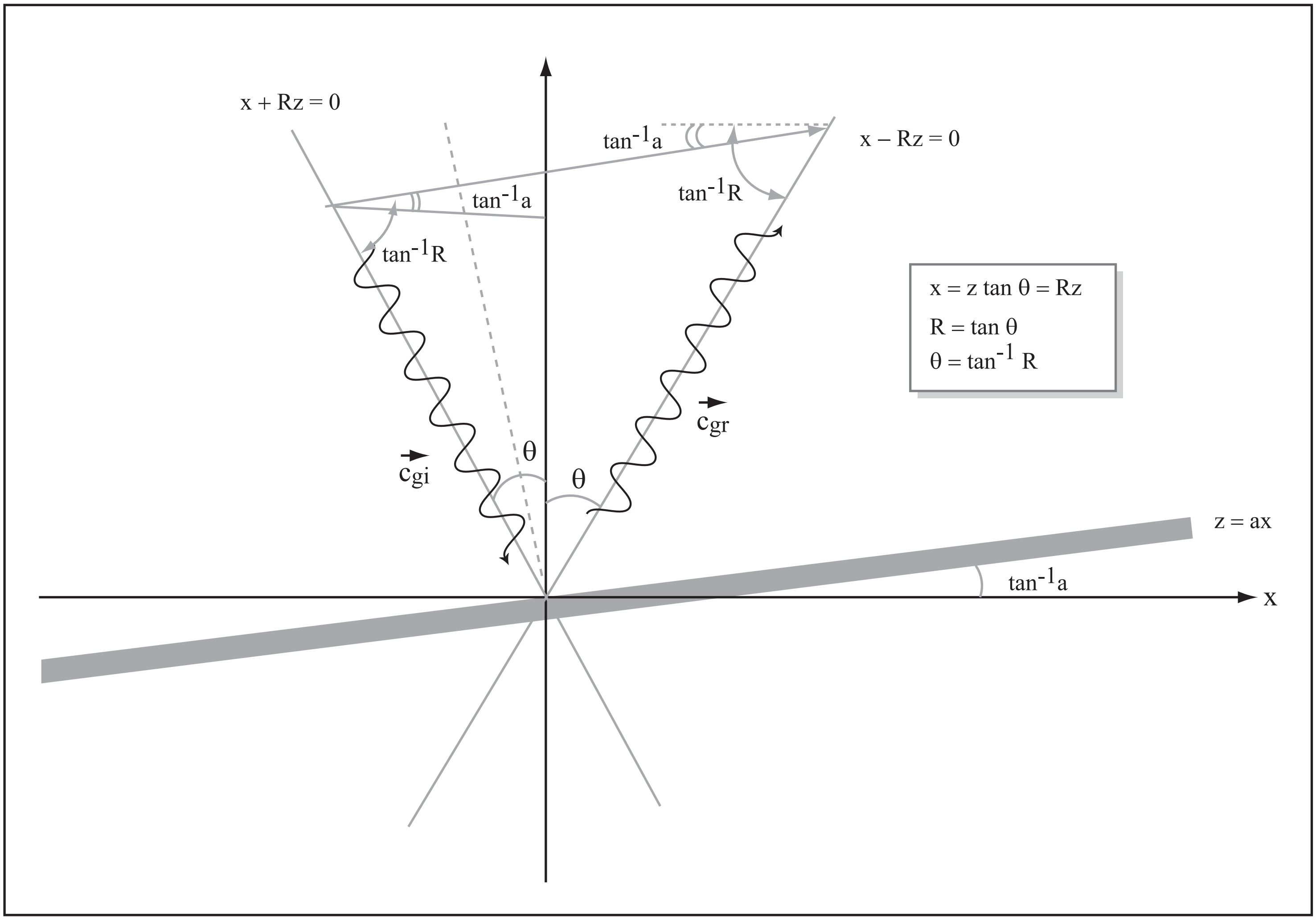
The wavelength shortens as a consequence of the reflection process.

Consider now the changes in group velocity $\vec{c}_g$

For the group velocity the component conserved is the component perpendicular to the wall as there cannot be an energy flux into the wall

$$c_{gi}|_{\perp wall} - c_{gr}|_{\perp wall} = 0$$

$$c_{gi}|_{\perp wall} = c_{gr}|_{\perp wall}$$



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$$|\vec{c}_{gi}|\sin(\tan^{-1}R + \tan^{-1}a) = |\vec{c}_{gr}|\sin(\tan^{-1}R - \tan^{-1}a)$$

or
$$\vec{c}_{gr} = -\vec{c}_{gi}\ \frac{(1+aR)}{(1-aR)}$$

While λ shortens in the reflection process, $\vec{c}_{gr}$ increases

Notice that if aR->1 the reflected $\vec{c}_{gr}$ is very large. What does this mean? It means that the bottom coincides with the outgoing characteristics: z = ax -> z = R^{-1}x.

As aR $\rightarrow$ 1, $\vec{c}_{gr}$ is very large, k_r is very large: the reflected wave is very small. The present inviscid analysis fails.

Rules for sloping bottom:

1. Angle θ of $\vec{c}_{gi}$, $\vec{c}_{gr}$ with the vertical must be the same.
2. The components $\perp$ to bottom must be equal

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