Lecture 22

22.1 Barotropic waves, no rotation

We'll spend the class talking about various *barotropic* waves. Baroclinic waves are *very*, *very*, important, but I'll be leaving them to others. Since we've already derived the shallow water equations, there is a nice path forward. Recall:

$$
\begin{aligned}
x: \quad & \frac{Du}{Dt} &=& -g\frac{\partial h}{\partial x} + fv & (22.1)\\
y: \quad & \frac{Dv}{Dt} &=& -g\frac{\partial h}{\partial y} - fu & (22.2)\\
h: \quad & \frac{Dh}{Dt} &+& h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 & (22.3)
\end{aligned}
$$

Let's begin by considering the case of no rotation

$$\Rightarrow fv = fu = 0 \tag{22.4}$$

Thus

$$
\begin{aligned}
x: \quad & \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &=& -g\frac{\partial h}{\partial x} & (22.5)\\
y: \quad & \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &=& -g\frac{\partial h}{\partial y} & (22.6)\\
h: \quad & \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} &+& h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 & (22.7)
\end{aligned}
$$

We want to find some wavy solutions to this set of equations that describe surface gravity waves. First we must linearize. Lots of ways to do it, let's do the "mean flow and perturbation" method:

$$h = \bar{h} + h' \qquad u = u' \qquad v = v' \tag{22.8}$$

Note that assuming $\bar{u} = 0$ and $\bar{v} = 0$ produced specific results. To make our life easier, we're going to assume the mean flow is static.

> **Aside**: We don't have to make this assumption. The $\bar{u}$, $\bar{v}$, and $\bar{h}$ terms are assumed to be a solution, so they will drop out . Consider the following:
>
> $$\frac{\partial a}{\partial t} + b\frac{\partial b}{\partial x} = 0 \tag{22.9}$$
>
> $$a = \bar{a} + a' \qquad b = \bar{b} + b' \tag{22.10}$$
>
> $$\frac{\partial \bar{a}}{\partial t} + \frac{\partial a'}{\partial t} + \bar{b}\frac{\partial \bar{b}}{\partial x} + \bar{b}\frac{\partial p'}{\partial x} + b'\frac{\partial \bar{b}}{\partial x} + b'\frac{\partial p'}{\partial x} = 0 \tag{22.11}$$
>
> $$\left(\frac{\partial \bar{a}}{\partial t} + \bar{b}\frac{\partial \bar{b}}{\partial x}\right) + \left(\frac{\partial a'}{\partial t} + \bar{b}\frac{\partial b'}{\partial x}\right) + \tag{22.12}$$

(Jim – The rest of the equations are cut off). You get different equations for different assumptions.

Back to the problem at hand. Substitution yields:

$$x: \qquad \frac{\partial u'}{\partial t} + u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} = -g\frac{\partial(h+h')}{\partial x} \tag{22.13}$$

$$y: \qquad \frac{\partial v'}{\partial t} + u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} = -g\frac{\partial(\bar{h}+h')}{\partial y} \tag{22.14}$$

$$h: \qquad \frac{\partial}{\partial t}(\bar{h}+h') + u'\frac{\partial}{\partial x}(\bar{h}+h') + v'\frac{\partial}{\partial y}(\bar{h}+h') + (\bar{h}+h')\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{22.15}$$

Simplify:

$$x: \qquad \frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \tag{22.16}$$

$$y: \qquad \frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} \tag{22.17}$$

$$h: \qquad \frac{\partial h'}{\partial t} + \bar{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{22.18}$$

These are the linearized shallow water equations with no rotation. They are three linear equations with three unknowns. We've got a couple of options here; we can put things in terms of a single variable now and then "assume a solution of the form"; or "assume a solution of the form" now and *then* put things in terms of a single variable. The former requires taking a bunch of extra derivatives with respect to time, so I'll do the latter... it's more obvious. Assume a solution of the form:

$$u' = Ue^{i(kx+ly-\omega t)} \tag{22.19}$$

$$v' = Ve^{i(kx+ly-\omega t)} \tag{22.20}$$

$$h' = He^{i(kx+ly-\omega t)} \tag{22.21}$$

Evaluating the derivatives we get:

$$\frac{\partial u'}{\partial t} = -i\omega Ue^{i(kx+ly-\omega t)} \tag{22.22}$$

$$\frac{\partial u'}{\partial x} = ikUe^{i(kx+ly-\omega t)} \tag{22.23}$$

$$\frac{\partial v'}{\partial t} = -i\omega Ve^{i(kx+ly-\omega t)} \tag{22.24}$$

$$\frac{\partial v'}{\partial y} = ilVe^{i(kx+ly-\omega t)} \tag{22.25}$$

$$\frac{\partial h'}{\partial t} = -i\omega He^{i(kx+ly-\omega t)} \tag{22.26}$$

$$\frac{\partial h'}{\partial x} = ikHe^{i(kx+ly-\omega t)} \tag{22.27}$$

$$\frac{\partial u'}{\partial y} = ilHe^{i(kx+ly-\omega t)} \tag{22.28}$$

$$\tag{22.29}$$

Substituting this into the three equations yields:

$$x: \qquad -i\omega Ue^{i(kx+ly-\omega t)} = -ikHge^{i(kx+ly-\omega t)} \tag{22.30}$$

$$y: \qquad -i\omega Ve^{i(kx+ly-\omega t)} = -ilHge^{i(kx+ly-\omega t)} \tag{22.31}$$

$$h: \qquad -i\omega He^{i(kx+ly-\omega t)} + \bar{h}(ikUe^{i(kx+ly-\omega t)} + ilUe^{i(kx+ly-\omega t)}) = 0 \tag{22.32}$$

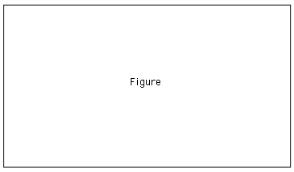

Figure 22.1: (`fig:Lec22NonRotGrv1`) Non-rotating gravity waves: Dispersion relationship for $c = \pm 1$.

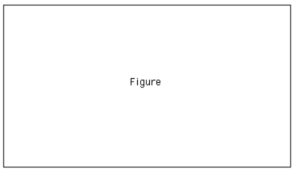

Figure 22.2: (`fig:Lec22NonRotGrv2`) Non-rotating gravity waves: Contours of ω.

All the i's and $e^{i(kx+ly-\omega t)}$'s go away

$$x: \qquad \omega U = gkH \tag{22.33}$$
$$y: \qquad \omega V = glH \tag{22.34}$$
$$h: \qquad -\omega H + \bar{h}kU + \bar{h}lU = 0 \tag{22.35}$$

Solve for U and V with the first two equations and substitute into the h equation:

$$-\omega H + \bar{h}K\left(\frac{gkH}{\omega}\right) + \bar{h}l\left(\frac{gkH}{\omega}\right) = 0 \tag{22.36}$$

H goes away:

$$-\omega^2 + \bar{h}gk^2 + \bar{h}gl^2 = 0 \tag{22.37}$$
$$\omega^2 = \bar{h}g(k^2 + l^2) \tag{22.38}$$
$$\omega = \pm\sqrt{\bar{h}g(k^2 + l^2)} \tag{22.39}$$
$$\omega = \pm\sqrt{\bar{h}gK^2} \qquad (K^2 = k^2 + l^2) \tag{22.40}$$
$$\omega = K\left(\pm\sqrt{\bar{h}g}\right) \tag{22.41}$$

This is the gravity wave dispersion relation. The phase speed is

$$c = \frac{\omega}{K} = \sqrt{\bar{h}g} \tag{22.42}$$

This $\sqrt{\bar{h}g}$ shows up so frequently it is defined to be c; $\omega = Kc$, where c can be plus or minus. Note that c_g and c_p are in the same direction.

Show Pix (see figures 22.1, 22.2, 22.3, and 22.4)

22.2 Shallow water equations with rotation

Let's put rotation back into the mix!

$$x: \qquad \frac{Du}{Dt} = -g\frac{\partial h}{\partial x} + fv \tag{22.43}$$

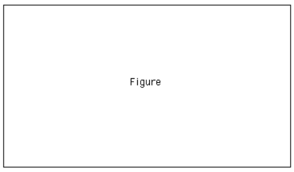

Figure 22.3: (`fig:Lec22NonRotGrv3`) Non-rotating gravity waves: Dispersion relationship for $l = 0$.

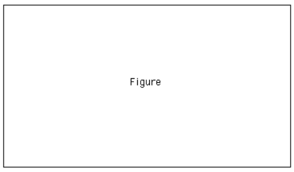

Figure 22.4: (`fig:Lec22NonRotGrv4`) Non-rotating gravity waves: Dispersion relationship for $l = .25$. Note that this looks dispersive if cut in the x or y direction. **(Jim – You may want to describe what you mean by this last sentence).**

$$y: \qquad \frac{Dv}{Dt} = -g\frac{\partial h}{\partial y} - fu \tag{22.44}$$

$$h: \qquad \frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{22.45}$$

Like last time, begin by linearizing, here using

$$h = \bar{h} + h' \qquad u = u' \qquad v = v' \tag{22.46}$$

Thus

$$x: \qquad \frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + \bar{f}v' \tag{22.47}$$

$$y: \qquad \frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - \bar{f}u' \tag{22.48}$$

$$h: \qquad \frac{\partial h'}{\partial t} + \bar{h}\left(\frac{\partial \omega}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{22.49}$$

Assume solutions of the form:

$$u' = Ue^{i(kx+ly-\omega t)} \tag{22.50}$$

$$v' = Ve^{i(kx+ly-\omega t)} \tag{22.51}$$

$$h' = He^{i(kx+ly-\omega t)} \tag{22.52}$$

$$\tag{22.53}$$

Substitute and solve:

$$\omega^3 - \omega\bar{f}^2 - \omega\bar{h}g(k^2 + l^2) = 0 \tag{22.54}$$

$$\omega(\omega^2 - \bar{f}^2 - \bar{h}g(k^2 + l^2)) = 0 \tag{22.55}$$

$$\tag{22.56}$$

This yields the following solutions

$$\omega = 0, \qquad \omega = \pm\sqrt{\bar{f}^2 + \bar{h}g(k^2 + l^2)} \qquad \text{or} \qquad \omega = \pm\sqrt{\bar{f}^2 + c^2K^2} \tag{22.57}$$

This is the dispersion relation for shallow water gravity waves with rotation (Poincare Waves). The $\omega = 0$ root represents geostrophic flow (no time dependence). The $\omega = \sqrt{\bar{f}^2 + c^2K^2}$ roots also have interesting properties:

- $\bar{f} = 0$ (no rotation) implies $\omega = cK$. That is, shallow water gravity waves with no rotation.
- Large K (small wavelengths) implies $\omega \approx cK$. That is, scales too small to feel rotation. We can quantify this as follows:

$$c^2K^2 \gg \bar{f}^2 \quad \Rightarrow \quad K^2 \gg \frac{\bar{f}^2}{c^2} \quad \Rightarrow \quad K \ll \frac{\bar{f}}{c} \tag{22.58}$$

  Recall $\frac{c}{f} = R_d$, the Rossby radius of deformation scale at which both gravity and rotation are important. Thus

$$K \gg \frac{1}{R_d} \quad \Rightarrow \quad \frac{2\pi}{\lambda} \gg \frac{1}{R_d} \quad \Rightarrow \quad \lambda \ll R_d \tag{22.59}$$

  The wavelengths are much much less than the Rossby radius of deformation, so don't feel rotation. Gravity dominates.

- Small K (large wavelengths) means $\omega \equiv \bar{f}$ which implies inertial motion!

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{f} = \frac{2\pi}{2\Omega \sin\phi} = \frac{\pi}{\Omega \sin\phi} \Rightarrow \frac{1}{2}\text{ a day!} \tag{22.60}$$

  Quantify in terms of R_d:

$$c^2K^2 \ll \bar{f}^2 \Rightarrow K^2 \ll \frac{\bar{f}^2}{c^2} \Rightarrow K \ll \frac{\bar{f}}{c} \tag{22.61}$$

$$\Rightarrow K \ll \frac{1}{R_d} \Rightarrow \frac{2\pi}{\lambda} \ll \frac{1}{R_d} \Rightarrow \lambda \gg R_d \tag{22.62}$$

  The wavelengths are much much greater than the Rossby radius of deformation. Rotation dominates. Recall that the cK bit of the dispersion diagram comes from:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv' \tag{22.63}$$

  if cK is small, implies $-g\frac{\partial h'}{\partial x}$ is small relative to fv':

$$\frac{\partial u'}{\partial t} = fv' \tag{22.64}$$

  This is just the inertial relationship!

We can see all all this stuff in the dispersion diagram:

Show PIX. $\omega = f(k, l)$. Note $\omega = 1$ at $(0, 0)$. (See figure 22.5).

Show PIX. $\omega = f(k)$. (See figure 22.6).

Because Poincare waves behave like inertia waves and gravity waves in the limits, they behave like a mixture of inertial and gravity waves in between. Thus, Poincare waves $\equiv$ inertia–gravity waves.

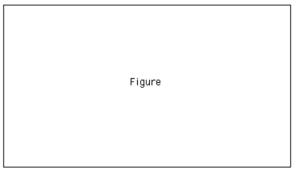

Figure 22.5: (`fig:Lec22RotGrv1`) Contours of ω.

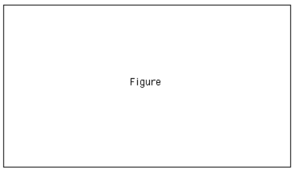

Figure 22.6: (`fig:Lec22RotGrv2`) Dispersion relation for $c = 1$, $f = 1$. Note the following: c_g in same direction as c_p. Red line is gravity wave. $\omega = 0$ is geostrophy. $\omega \geq \bar{f}$.

22.3 Kelvin Wave

The Kelvin wave is a wave that needs to lean against a lateral boundary like a coastline or the equator. It leans to the right in the NH. See figure 22.7. It is wavelike in the direction of travel and decays as you move away from the boundary. Let's see what we can learn about this wave from the linearized shallow water equations:

$$
\begin{aligned}
x: \qquad \frac{\partial u'}{\partial t} &= -g\frac{\partial h'}{\partial x} + \bar{f}v' && (22.65)\\
y: \qquad \frac{\partial v'}{\partial t} &= -g\frac{\partial h'}{\partial y} - \bar{f}u' && (22.66)\\
h: \qquad \frac{\partial h'}{\partial t} &+ \bar{h}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 && (22.67)
\end{aligned}
$$

The boundary conditions tell us that $v(0) = 0$, where $v(0)$ is the velocity normal to the wall at the wall. Let's make our lines easy and state:

$$v = 0 \Rightarrow v' = 0 \qquad \text{always} \tag{22.68}$$

This simplifies our equations:

$$
\begin{aligned}
x: \qquad \frac{\partial u'}{\partial t} &= -g\frac{\partial h'}{\partial x} && (22.69)\\
y: \qquad fu' &= -g\frac{\partial h'}{\partial y} && (22.70)
\end{aligned}
$$

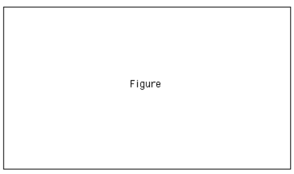

Figure 22.7: (`fig:Lec22KelvinWave`) Kelvin wave.

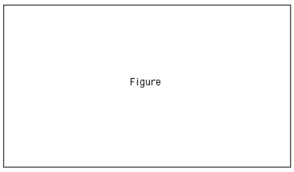

Figure 22.8: (`fig:Lec22NonDispWave`) A non-dispersive wave.

$$h: \qquad \frac{\partial h'}{\partial t} \quad + \quad \bar{h}\left(\frac{\partial u'}{\partial x}\right) = 0 \tag{22.71}$$

Let's get a single equation for u' instead of assuming solutions for both u' and h', just to show it can be done. Can get a $\frac{\partial h'}{\partial t}$ by taking $\frac{\partial}{\partial t}$ of the x equation.

$$\frac{\partial}{\partial t}x: \qquad \frac{\partial^2 u'}{\partial t^2} = -g\frac{\partial}{\partial x}\frac{\partial h'}{\partial t} \tag{22.72}$$

Substitute in $\frac{\partial h'}{\partial t} = -\bar{h}\frac{\partial u'}{\partial x}$ and we get

$$\frac{\partial^2 u'}{\partial t^2} = g\bar{h}\frac{\partial^2 u'}{\partial x^2} \tag{22.73}$$

Now we have 1 equation and 1 unknown. Assume a solution of the form

$$u' = U(y)e^{i(kx-\omega t)} \tag{22.74}$$

We make the y dependent bit the way it is because we can and we will have to. Substitute and get

$$\omega = \pm ck \qquad c = \sqrt{g\bar{h}} \tag{22.75}$$

Kelvin wave has a dispersion relation identical to shallow water gravity waves: same speed and non-dispersive (see figure 22.8).

Now let's get at the form of the Kelvin wave. We have two solutions to ω so by linear superposition

$$\begin{aligned} u' &= U_1 e^{i(kx+\omega t)} + U_2 e^{i(kx-\omega t)} && (22.76)\\ \Rightarrow \quad u' &= U_1 e^{ik(x+ct)} + U_2 e^{ik(x-ct)} && (22.77) \end{aligned}$$

Plug solution into the x equation:

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} \tag{22.78}$$

and integrate to get

$$h' = \sqrt{\frac{\bar{h}}{g}}U_1 e^{ik(x+ct)} - \sqrt{\frac{\bar{h}}{g}}U_2 e^{ik(x-ct)} \tag{22.79}$$

Nice, but what are U_1 and U_2? To get U_1 and U_2 plug h' into y equation

$$fu' = -g\frac{\partial h'}{\partial y} \tag{22.80}$$

Note geostrophic! Equate the U_1 and U_2 bits

$$\frac{\partial U_1}{\partial y} = \frac{f}{c}U_1 \quad \Rightarrow \quad U_1 = U_{10}e^{\frac{f}{c}y} \tag{22.81}$$

$$\frac{\partial U_2}{\partial y} = -\frac{f}{c}U_2 \quad \Rightarrow \quad U_2 = U_{20}e^{-\frac{f}{c}y} \tag{22.82}$$

The first solution is unbounded and therefore unphysical while the second solution decays with y and is therefore acceptable:

$$u' = U_0 e^{-\frac{f}{c}y} e^{ik(x-ct)} \tag{22.83}$$

Now to get form of h'; plug U_2 into h' equation

$$h' = -\sqrt{\frac{\bar{h}}{g}} U_0 e^{-\frac{f}{c}y} e^{ik(x-ct)} \tag{22.84}$$

Recall $R_d = \frac{c}{f}$ so

$$u' = U_0 e^{-\frac{y}{R_d}} e^{ik(x-ct)} \tag{22.85}$$

$$v' = 0 \tag{22.86}$$

$$h' = \sqrt{\frac{\bar{h}}{g}} U_0 e^{-\frac{f}{c}y} e^{ik(x-ct)} \tag{22.87}$$

Note R_d sets the spatial scale in the y direction.

ECCO movie to show Kelvin waves.

Delayed-action oscillator model of ENSO.

- Westerly wind burst fires off a downwelling Kelvin wave and an upwelling Rossby wave.
- West Pacific so deep the Rossby wave has little impact.
- East Pacific is shallow and downwelling Kelvin deepens the thermocline.
- West Pacific Rossby waves hit boundary and reflect back as upwelling Kelvin waves which deepen the thermocline in the coast.
- Etc.