

Lecture 16

16.1 Administrative

- Exam stuff:
 - 2D versus 3D in Problem 1
 - make sure you know stuff
 - please make visit
- Circulation:

$$\Gamma_a = \iint_A (\boldsymbol{\omega} \cdot 2\boldsymbol{\Omega}) \cdot d\mathbf{A}. \quad (16.1)$$

What is the integral form? Since $\Gamma = \oint \mathbf{u} \cdot d\mathbf{s} = \iint_A \boldsymbol{\omega} \cdot d\mathbf{A}$ in inertial frame and

$$\mathbf{u}_f = \mathbf{u}_r + \boldsymbol{\Omega} \times \mathbf{r} \quad (16.2)$$

$$\Rightarrow \mathbf{u}_r = \mathbf{u}_f - \boldsymbol{\Omega} \times \mathbf{r} \quad (16.3)$$

then

$$\oint \mathbf{u} \cdot d\mathbf{s} = \oint (\mathbf{u} - \boldsymbol{\Omega} \times \mathbf{r}) d\mathbf{s} \quad (16.4)$$

16.2 Eddy diffusivity

Start today by letting you in on one of the fdrty little secrets of GFD... eddy diffusivity. For incompressible N-S, the friction terms had the form

$$F_x = \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2} + \nu \frac{\partial^2 u}{\partial z^2} \quad (16.5)$$

Last lecture we argued that the first two terms are small compared to the third because of the length scales in the denominator. True enough. But in the modeling world we have the problem

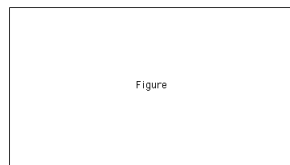


Figure 16.1: (fig:Lec16TempVelSnapshot1) Temperature and currents at high resolution

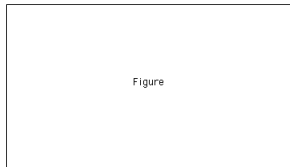


Figure 16.2: (fig:Lec16TempVelSnapshot2) Temperature and currents at low resolution

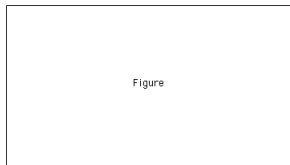


Figure 16.3: (fig:Lec16Mixing) Comment of isopycnal mixing: Really, ν is huge along constant density surfaces, and tiny across constant density surfaces. Because constant density surfaces slope, end up with large ν_H and ν_V .

of unresolved scales. There is all kinds of stuff happening on spatial scales smaller than our grid spacing (see figures 16.1 and 16.2). So, to model the *impact* of these unresolved scales, we put in an artificial viscosity:

$$F_x = \nu_H \frac{\partial^2 u}{\partial x^2} + \nu_H \frac{\partial^2 u}{\partial y^2} + \nu_V \frac{\partial^2 u}{\partial z^2} \quad (16.6)$$

where ν_H and ν_V differ by 3 or 4 orders of magnitude, and ν_H and ν differ by ~ 7 orders of magnitude (see figure 16.3 for note on isopycnal mixing).

16.3 Balances in the simplified momentum equations

We're going to spend the next two or three lectures looking at "balances" in the simplified momentum equations. Ignoring viscosity for now:

$$x : \quad \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (16.7)$$

$$y : \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \quad (16.8)$$

$$z : \quad 0 = -\frac{\partial p}{\partial z} - pg \quad (16.9)$$

The assumptions that went into deriving these equations are no viscosity, incompressible, no curvature terms, no vertical coriolis terms, and no vertical rates of change (these last two correspond to hydrostatic balance).

16.4 f -plane

Note that $f = 2\Omega \sin \phi$ is a function of latitude. If the time-scales are long and/or the spatial scales are large, then the latitudinal dependence is important. But we can often assume that the coriolis parameter is constant:

$$f_0 = 2\Omega \sin \phi_0 \quad (16.10)$$

where ϕ_0 is a reference latitude. This is the f -plane approximation (**Q1**).

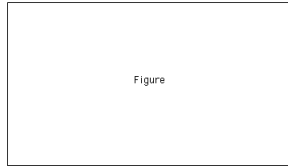


Figure 16.4: (**fig:Lec16BetaPlane**) Relating small changes in latitude to small changes in y .

16.5 β -plane

If latitudinal changes are likely to be important, but we can restrict ourselves to an area near a reference latitude, then we can allow the coriolis parameter to vary linearly with y . Perform a Taylor expansion about some reference latitude:

$$f = f(\phi) = 2\Omega \sin \phi \quad (16.11)$$

$$f(\phi_0 + \Delta\phi) = f(\phi_0) + f'(\phi_0)\Delta\phi \quad (16.12)$$

$$= f_0 + 2\Omega \cos \phi_0 \Delta\phi \quad (16.13)$$

Noting figure 16.4, we get:

$$f = f_0 + 2\Omega \cos \phi_0 \frac{y}{r} \quad (16.14)$$

$$f = f_0 + \frac{2\Omega \cos \phi_0}{r} y \quad (16.15)$$

$$f = f_0 + \beta y \quad (16.16)$$

Where $\beta = \frac{2\Omega \cos \phi_0}{r}$. This is the β -plane approximation.

16.6 Recap of approximations to f

Back to our simplified momentum equations.

$$x : \quad \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (16.17)$$

$$y : \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \quad (16.18)$$

$$z : \quad 0 = -\frac{\partial p}{\partial z} - pg \quad (16.19)$$

$$(16.20)$$

Here f could be

$$f = 2\Omega \sin \phi \quad (16.21)$$

$$f = f_0 + \rho y \quad (16.22)$$

$$f = f_0 \quad (16.23)$$

We'll stick with f_0 for a while. There are many many things to say about these equations. A *huge* amount of observed geophysical flows can be described by *further* simplifications of these already simplified equations.

16.7 Inertial circles

We've already talked about one simplification. In local cylindrical coordinates:

$$\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \Delta)u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + fu_\theta \quad (16.24)$$

Inertial circles have no u_r , only u_θ , and there is no pressure gradient force. We are left with the "balance"

$$\frac{-u_\theta^2}{r} = fu_\theta \quad (16.25)$$

This describes the observed inertial circles, and gave us the period of $\frac{1}{2}$ day. Remember that? **(Q2)**. Strictly, I probably shouldn't call this a balance. Balances are between forces, this is an $F = ma$ expression.

16.8 Geostrophic balance

This next simplification really does result in a proper balance.

$$x : \quad \frac{Du}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (16.26)$$

$$y : \quad \frac{Dv}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - fu \quad (16.27)$$

$$(16.28)$$

The easy way out is to say we want to look at the time-invariant flow, so $\frac{Du}{Dt} = 0$ by definition. This leaves us with:

$$x : \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = fv \quad (16.29)$$

$$y : \quad \frac{1}{\rho} \frac{\partial p}{\partial y} = -fu \quad (16.30)$$

$$(16.31)$$

This is a balance between the pressure gradient force and the coriolis force. It is called the *geostrophic balance* or the concept of *geostrophy*. Before saying more about the balance, lets get a feel for the situations in which we can expect geostrophic flow. Write down the simplified x -momentum equation:

$$x : \quad \frac{\partial u}{\partial t} + (\mathbf{u} \cdot \nabla)u = \frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (16.32)$$

$$\frac{U}{T} = \frac{U}{L/U} = \frac{U^2}{L} \quad \frac{U^2}{L} \quad fU \quad (16.33)$$

When is it okay to ignore the LHS? The terms of the LHS are order $\frac{u^2}{L}$, the coriolis term is fU . To ignore LHS, we need

$$\frac{U^2}{L} \ll fU \quad (16.34)$$

$$\frac{U}{fL} \ll 1 \quad (\mathbf{Q4}) \quad (16.35)$$

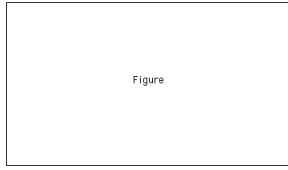


Figure 16.5: (fig:Lec16GeostrophicForceBalance) Geostrophic force balance (**Q3**).

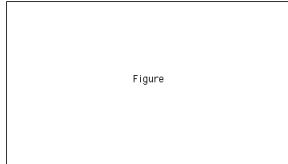


Figure 16.6: (fig:Lec16EqPoleTemp) Equator-pole temperature difference and its importance to geostrophy.

But

$$\frac{U}{fL} = R_0 \quad (16.36)$$

Remember the Rossby number? So we can only ignore the LHS for $R_0 \ll 1 \Rightarrow$ impact of rotation is important relative to time rates of velocity. $R_0 \ll 1$ for mid-latitude, large scale features.

Back to geostrophy: Balance between the pressure gradient force and the coriolis force. Imagine a pressure gradient as is seen in figure 16.5. This will drive a flow in the $+y$ direction. In the northern hemisphere, this flow will be deflected to the right, and will continue to deflect until the pressure gradient force balances the coriolis force. This will happen when the flow is parallel to lines of constant pressure. This same result is seen in the math:

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} + fv \quad (16.37)$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial x} - fu \quad (16.38)$$

$$\Rightarrow v = \frac{1}{f\rho} \frac{\partial p}{\partial x} \quad (16.39)$$

$$\Rightarrow u = -\frac{1}{f\rho} \frac{\partial p}{\partial y} \quad (16.40)$$

$$\Rightarrow \mathbf{u}_g = \frac{1}{f\rho} \mathbf{k} \times \nabla p \quad (16.41)$$

Here \mathbf{u}_g = geostrophic velocity. Note that this equation says \mathbf{u}_g is perpendicular to the pressure gradient. In the atmosphere, you have heating at the equator and cooling at the poles (see figure

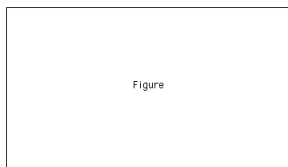


Figure 16.7: (fig:Lec16HorizontalSlice) Plane view of horizontal slice from 16.6.

16.6). So the height of a pressure surface is lower at the poles than at the equator. Take a horizontal slice through these pressure surfaces and look at the plan view (see figure 16.7). The horizontal pressure decreases as we go from the equator to the pole, so the resulting geostrophic flow is westerly, explaining our prevailing winds in the Northern Hemisphere. Is this what we see in the observation? Yes, but before I show you any observations, let me talk a bit about pressure as a vertical coordinate. Consider:

$$\left. \frac{\partial p}{\partial x} \right|_z \quad (16.42)$$

From hydrostatic, know $p = p(z)$, hence for constant z we must use the chain rule

$$\left. \frac{\partial p}{\partial x} \right|_z = - \left. \frac{\partial p}{\partial z} \right|_x \frac{\partial z}{\partial x} \Big|_p + \frac{\partial p}{\partial \rho} \frac{\partial \rho}{\partial x} \quad (16.43)$$

Note that the negative sign is because pressure decreases at z increases. Also, we can ignore the last term if we have a constant density. Next, the hydrostatic balance says:

$$\frac{\partial p}{\partial z} = -\rho g \quad (16.44)$$

Substitute into $\frac{\partial p}{\partial x}$ equation: $\frac{\partial p}{\partial x} = \rho g \frac{\partial z}{\partial x} \Big|_p$:

$$x : \quad \frac{1}{\rho} \frac{\partial p}{\partial x} = f v \Rightarrow \frac{1}{\rho} \left(\rho g \frac{\partial z}{\partial x} \right) = f v \quad (16.45)$$

$$\Rightarrow v = \frac{g}{f} \left. \frac{\partial z}{\partial x} \right|_p \quad (16.46)$$

$$y : \quad \frac{1}{\rho} \frac{\partial p}{\partial y} = -f u \Rightarrow \frac{1}{\rho} \left(\rho g \frac{\partial z}{\partial y} \right) = -f u \quad (16.47)$$

$$\Rightarrow u = -\frac{g}{f} \left. \frac{\partial z}{\partial y} \right|_p \quad (16.48)$$

Thus,

$$\mathbf{u}_g = \frac{g}{f} \mathbf{k} \times \nabla z \Big|_p \quad (16.49)$$

$$\mathbf{u}_g = \frac{1}{f} \mathbf{k} \times \nabla p \Big|_z \quad (16.50)$$

For the first equation, p is the independent variable, and z is the dependent variable. Instead of z , which can be confusing, call it h for height: $\mathbf{u}_g = \frac{g}{f} \mathbf{k} \times \nabla h \Big|_p$.

Show observations of geostrophic balance.

See figures 16.8 through 16.14.

16.9 Ageostrophic flow

So the observed flow is largely geostrophic, but there are areas where there is significant ageostrophic flow. What could be causing this? Well, lots of stuff... all the stuff we've left out. But concentrate on the simplified equations we started with:

$$x : \quad \frac{\partial u}{\partial t} + u \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial x} + f v \quad (16.51)$$

$$y : \quad \frac{\partial v}{\partial t} + v \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) = -\frac{1}{\rho} \frac{\partial p}{\partial y} - f u \quad (16.52)$$

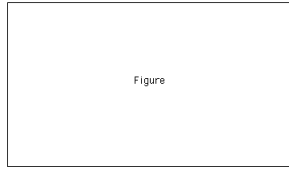


Figure 16.8: (`fig:Lec16HeightSurfaces`) Height p like, $\log p$ if z like. So we do see higher pressure South and lower pressure North (follow a constant pressure, horizon, line).

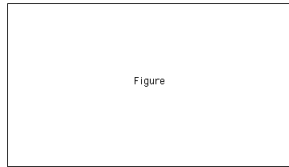


Figure 16.9: (`fig:Lec16Observed1`) Case 1: Observed winds. Note contours are height of 500 mb surface, units are m/s.

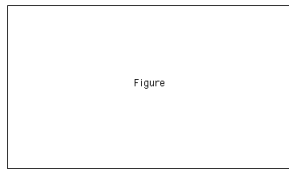


Figure 16.10: (`fig:Lec16Geostrophic1`) Case 1: Geostrophic winds.

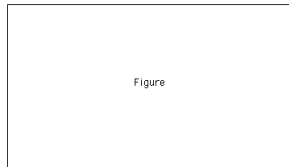


Figure 16.11: (`fig:Lec16Ageostrophic1`) Case 1: Ageostrophic winds

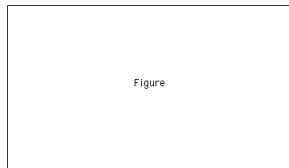


Figure 16.12: (`fig:Lec16Observed2`) Case 2: Observed wind

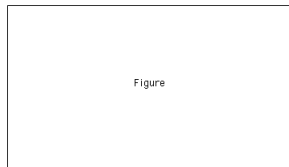


Figure 16.13: (`fig:Lec16Geostrophic2`) Case 2: Geostrophic winds.

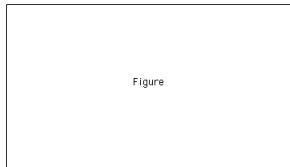


Figure 16.14: (fig:Lec16Ageostrophic2) Case 3: Ageostrophic winds.

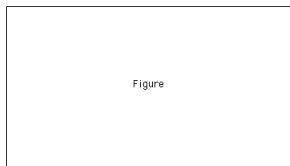


Figure 16.15: (fig:Lec16CylindricalCoordinates) Gradient wind is best thought of in terms of gradient wind.

We will next talk about the bit of the ageostrophic wind that comes from the $\frac{\partial}{\partial t}$, and the bit that comes from the $\mathbf{u} \cdot \nabla \mathbf{u}$ term.

16.10 Gradient Wind

Where the geostrophic wind is a two-way balance between the coriolis force and the pressure gradient force, the gradient wind is a three-way “balance” between the coriolis force, the pressure gradient force and the centrifugal force. Because we’re dealing with centrifugal force, it will be easiest to think of things in cylindrical coordinates (see figure 16.15).

$$r : \quad \frac{Du}{Dt} r = \frac{1}{\rho} \frac{\partial p}{\partial r} + fu_\theta \quad (16.53)$$

Expanding the LHS gives

$$\frac{\partial u_r}{\partial t} + \mathbf{u} \cdot \nabla u_r - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + fu_\theta \quad (16.54)$$

Assume steady and neglect nonlinear advectons

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + fu_\theta \quad (16.55)$$

or in terms of forces

$$-\frac{1}{\rho} \frac{\partial p}{\partial r} + fu_\theta + \frac{u_\theta^2}{r} = 0 \quad (16.56)$$

The second terms is the Coriolis bit which (see figure 16.15) in the direction of $+r$ which means that the term is greater than 0. The third terms is the centrifugal bit. It is always positive and in direction of $+r$. The first term, which here must balance the last two terms, is the pressure gradient bit. To balance $\frac{\partial p}{\partial r}$ must be greater than zero to that there is increasing pressure with increasing r .

How do we expect the gradient flow to compare with the geostrophic flow? See figures 16.16 and 16.17. The centrifugal force effectively reduces the pressure gradient force, which means that

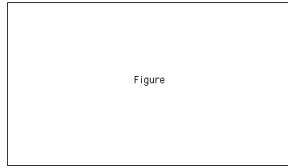


Figure 16.16: (fig:Lec16Anticyclone) Gradient wind balance in an anticyclone.

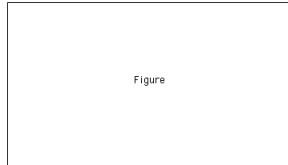


Figure 16.17: (fig:Lec16Cyclone) Gradient wind balance in a Cyclone.

the coriolis force in the gradient flow is less than for geostrophy, so u_θ in the gradient flow is less than the geostrophic u_θ (for flow around a low).

Look for this in observations.

Note that the ageostrophic flow is along pressure (or height lines).

Generically,

$$\frac{u_\theta^2}{r} - fu_\theta = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad (16.57)$$

and from geostrophy; $-\frac{1}{\rho} \frac{\partial p}{\partial r} = -fu_\theta$

16.11 Reading

None assigned in this lecture.