12.520 Lecture Notes 23

Navier-Stokes Equation – dimensional form

$$-\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i = \rho \frac{Dv_i}{Dt}$$

Assume:

Characteristic velocity v_0

Characteristic length L

Characteristic stress $\eta v_0 / L$

Characteristic time v_0 / L

Choose non-dimensional variables

$$v' = v / v_0 \quad x' = x / L \quad p' = \frac{L}{\eta v_0} p \quad t' = \frac{v_0}{L} t$$

or

$$v = v' v_0 \quad x = x' L \quad p = \frac{\eta v_0}{L} p' \quad t' = \frac{L}{v_0} t'$$

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x'} \quad f' = \frac{L^2}{\eta v_0} f \quad \frac{\partial}{\partial t} = \frac{v_0}{L} \frac{\partial}{\partial t'}$$

Substitute into Navier-Stokes equation

$$-\frac{1}{L} \frac{\eta v_0}{L} \frac{\partial p'}{\partial x_i'} + \frac{1}{L^2} v_0 \eta \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'} + \frac{1}{L^2} v_0 \eta f_i' = \rho v_0 \frac{v_0}{L} \frac{Dv_i'}{Dt'}$$

or

$$-\frac{\partial p'}{\partial x_i'} + \frac{\partial^2 v_i'}{\partial x_j' \partial x_j'} + f_i' = \frac{\rho v_0 L}{\eta} \frac{Dv_i'}{Dt'}$$

where $\dfrac{\rho v_0 L}{\eta} = \text{Re}$

$\Rightarrow$ Re gives importance of inertial terms relative to viscous terms.

$\text{Re} \ll 1$ viscous forces $\approx$ balance

acceleration negligible

$\text{Re} \gg 1$ inertia dominates

Note: in dimensionless form, Re is the only parameter in the Navier-Stokes equation.

$\Rightarrow$ for given geometry (boundary conditions) ALL equivalent

(non-dimensional) problems at same Re give same result!

Examples:

1. Low Reynolds number flow past a cylinder.

   $\text{Re} \ll 1$ Symmetry, like in the sphere problem.

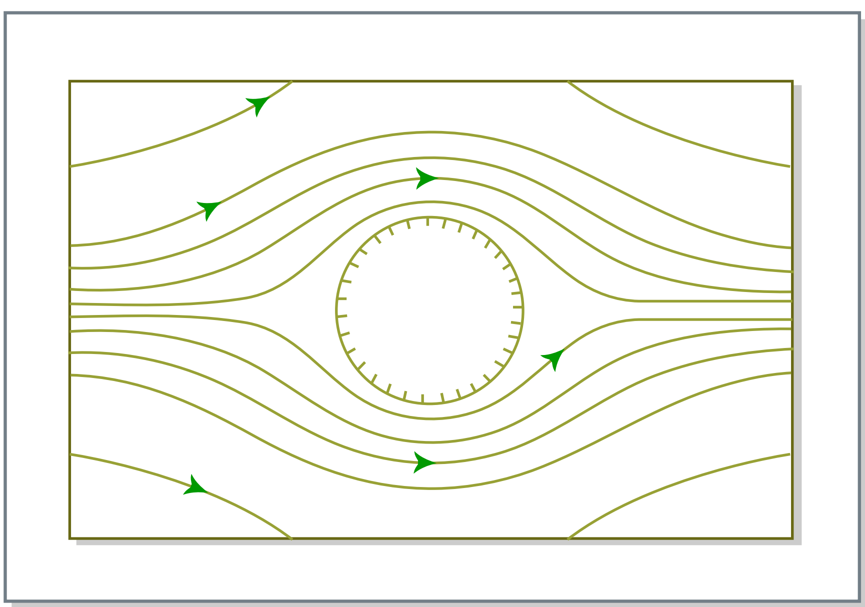

Figure 23.1

Figure by MIT OCW.

2. $\mathrm{Re} = 10 \quad \frac{\partial v_i}{\partial t} = 0$ (steady) $\quad v_j \frac{\partial v_i}{\partial x_j} \neq 0$

Asymmetry; eddies in wade.

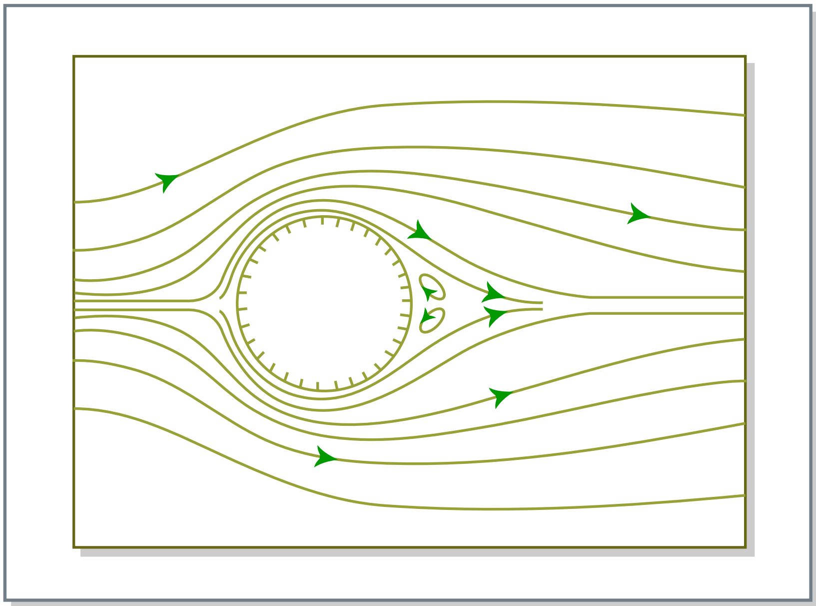

Figure 23.2
Figure by MIT OCW.

The figure below is experimental.

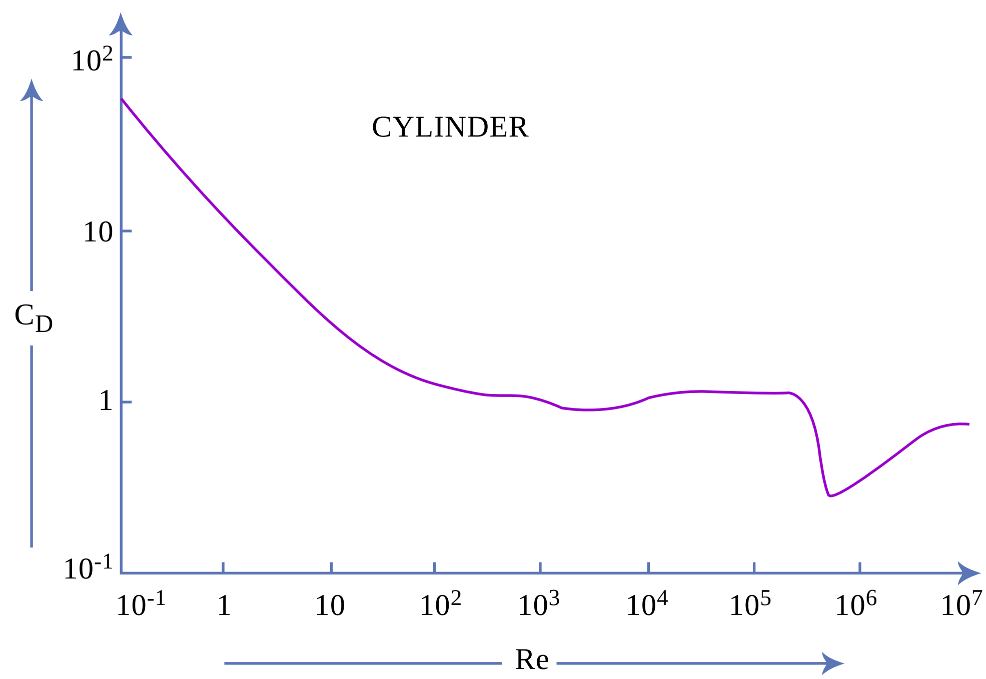

Figure 23.3. Variation of drag coefficient with Reynolds number for circular cylinder.

Figure 23.3
Figure by MIT OCW.

$Re \lesssim 1$	$C_D \sim 1 / Re$	$D \sim V$
$10^2 \lesssim Re \lesssim 3 \cdot 10^5$	$C_D \sim$ const.	$D \sim V^2$
$Re \sim 3 \cdot 10^5$	big drop in C_D!	

Recall

Earth's mantle: $Re \sim 10^{-19}$

canoe: $Re \sim 2 \cdot 10^5$

Summary:

For low Re, inertia not important

Navier-Stokes equation linear

"simple" results (analytic theory)

For high Re, inertia important

Navier-Stokes equation nonlinear

time dependent

complicated – experimental approach $\rightarrow$ empirical