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12.510 Introduction to Seismology  
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## **12.510 Introduction to Seismology**

**Date:** 04/24/2008

**Topic:** Anisotropy

**Organization:**

1. Basic Concepts
2. Hexagonal System
3. Elastic Moduli Tensor
4. Equation of Motion

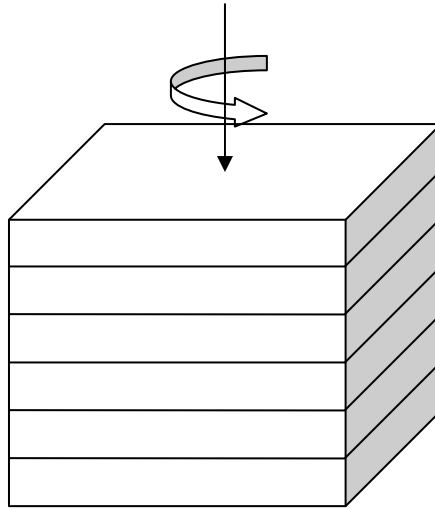
### **I. Basic Concepts**

So far, the course has considered wave propagation only in isotropic media. In isotropic media, properties are the same in all directions. For seismology, this means that wave speed does not depend on direction and raypaths are perpendicular to wavefronts.

Sometimes, however, it is useful to consider wave propagation in anisotropic media. Studying mantle flow at depth, for example, is aided by understanding how seismic waves propagate through anisotropic layers of aligned olivine crystals. In this type of media, properties depend on direction. This means that wave speed depends on direction and raypaths are not perpendicular to wavefronts.

### **II. Hexagonal System**

A simple example of an anisotropic system that is very useful in seismology is the hexagonal system. This system is simple because properties vary with direction in only two of the three planes defined by a set of orthogonal axes. Graphically, this looks something like:



Notice how everything looks the same if you look at the plane perpendicular to the arrow. If you look at the other two planes, though, there is directionality.

In the orientation shown, the anisotropy is called vertical transverse anisotropy. If the picture were tilted  $90^\circ$ , it would be called horizontal transverse anisotropy or azimuthal anisotropy.

### III. Elastic Moduli Tensor

Hooke's Law relates stresses to strains:

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

In isotropic media, the elastic moduli tensor reduces to two terms only,  $\lambda$  and  $\mu$ . In anisotropic media, there are more than two terms.

Since it is confusing to write out all the terms of the elastic moduli tensor, a simplified convention is often used. In this convention, the 81 terms in the tensor are written using only a  $6 \times 6$  matrix. This is possible because the symmetry of the stress and strain tensors and constraints from strain energy mean that there are only 21 terms in the tensor are independent.

The following notation rules link each element in the 81-element tensor to an element in the 6 x 6 matrix:

$$\begin{aligned}
 c_{ijkl} &\rightarrow C_{IJ} \\
 11 &\rightarrow 1 \\
 22 &\rightarrow 2 \\
 33 &\rightarrow 3 \\
 23 &\rightarrow 4 \\
 13 &\rightarrow 5 \\
 12 &\rightarrow 6
 \end{aligned}$$

For example, the element  $c_{1122}$  becomes  $C_{12}$ . In isotropic media, the matrix is:

$$C_{IJ} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

For anisotropic media it will be more complicated.

In global seismology, the elements in the matrix are often labeled a certain way using letters. This is called the Love convention:

$$C_{IJ} = \begin{pmatrix} A & A - 2N & F & 0 & 0 & 0 \\ A - 2N & A & F & 0 & 0 & 0 \\ F & F & C & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{pmatrix}$$

## IV. Equation of Motion

The equation of motion is:

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_i} \sigma_{ij}$$

Using Hooke's Law, this can be rewritten as:

$$\rho \ddot{u}_i = c_{ijkl} \partial_j \partial_l u_k$$

By defining the displacement as

$$\vec{u} = \vec{g} e^{iw(t - \vec{s} \cdot \vec{x})}$$

where

$\vec{g}$  is the polarization direction

$\vec{s}$  is the slowness vector

$\vec{x}$  is the position vector

the equation can again be rewritten as:

$$\rho g_i = g_k c_{ijkl} s_j s_l$$

or

$$(c_{ijkl} s_j s_l - \rho \delta_{ik}) g_k = 0$$

Notice that  $i$  is the only free parameter. Finally, normalize the elastic moduli tensor by the density

$$\frac{c_{ijkl}}{\rho} = \Gamma_{ijkl}$$

and define the Christoffel Matrix

$$M_{ik} = \Gamma_{ijkl} \hat{s}_j \hat{s}_l$$

to arrive at a new form of the equation of motion:

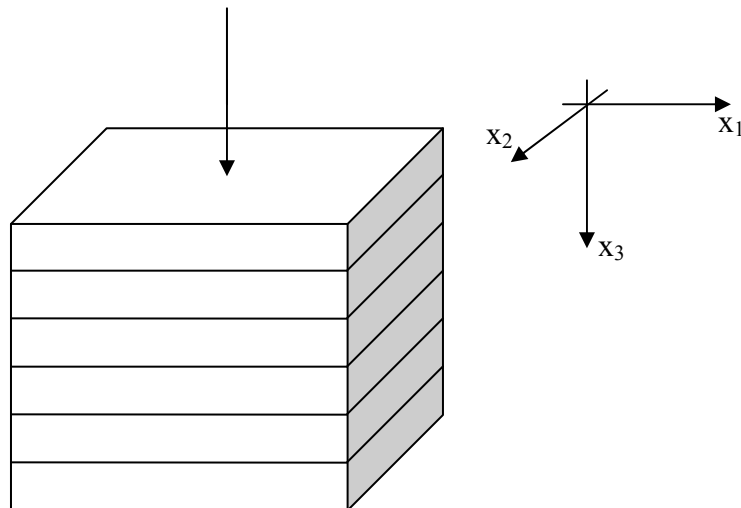
$$(\Gamma_{ijkl}\hat{s}_j\hat{s}_l - c^2\delta_{ik})g_k = 0$$

This equation is called the Christoffel Equation and  $c$  is the phase velocity. Notice that this equation is still general, meaning that nothing about anisotropy has been introduced yet. Anisotropy will be introduced via the elastic moduli tensor in the Christoffel Matrix. Also notice that the equation is an eigenvalue problem: solving it will yield three wavespeeds associated with three different directions.

As examples of how to solve the equation, consider two cases involving vertical transverse isotropy.

Case 1: Propagation in the Direction of Symmetry

This is the case shown in the picture of hexagonal symmetry above, with the arrow representing the raypath.



Assume the displacement is given by:

$$u_i = A_i \sin \left\{ \omega \left( \frac{x_3}{c} - t \right) \right\}$$

Then the Christoffel Matrix is given by

$$M_{ik} = \frac{1}{\rho} \begin{pmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & C \end{pmatrix}$$

and the Christoffel Equation is

$$\begin{bmatrix} \frac{L}{\rho} - c^2 & 0 & 0 \\ 0 & \frac{L}{\rho} - c^2 & 0 \\ 0 & 0 & \frac{C}{\rho} - c^2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = 0$$

Solving the equations yields three solutions:

$$\begin{aligned} c_1 &= \sqrt{\frac{L}{\rho}} & g_1 &= (1,0,0) \\ c_2 &= \sqrt{\frac{L}{\rho}} & g_2 &= (0,1,0) \\ c_3 &= \sqrt{\frac{C}{\rho}} & g_3 &= (0,0,1) \end{aligned}$$

Notice that since  $c_1$  and  $c_2$  are the same, it is impossible to tell that there is anisotropy in the transverse direction.

## Case 2: Propagation Orthogonal to the Direction of Symmetry

In this case, consider the propagation of a ray in the  $x_1$  direction in the picture above. Assume the displacement is given by:

$$u_i = B_i \sin \left\{ \omega \left( \frac{x_1}{c} - t \right) \right\}$$

Then the Christoffel Matrix is given by

$$M_{ik} = \frac{1}{\rho} \begin{pmatrix} A & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & L \end{pmatrix}$$

and the Christoffel Equation is

$$\begin{bmatrix} \frac{A}{\rho} - c^2 & 0 & 0 \\ 0 & \frac{N}{\rho} - c^2 & 0 \\ 0 & 0 & \frac{L}{\rho} - c^2 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = 0$$

Solving the equations yields three solutions:

$$\begin{aligned} c_1 &= \sqrt{\frac{A}{\rho}} & g_1 &= (1,0,0) \\ c_2 &= \sqrt{\frac{N}{\rho}} & g_2 &= (0,1,0) \\ c_3 &= \sqrt{\frac{L}{\rho}} & g_3 &= (0,0,1) \end{aligned}$$

Notice that since  $c_1$  and  $c_2$  are the same, it is impossible to tell that there is anisotropy in the transverse direction.