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12.510 Introduction to Seismology  
Spring 2008

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# 12.510 Introduction to Seismology

## Normal Modes (Free Oscillations) and Anisotropy

April 23, 2008

### Subdisciplines of Seismology

Seismology can be decomposed into three representative subdisciplines: body waves, surface waves, and normal modes of free oscillation. Technically, these domains form a continuum, each pertaining to particular frequency bands, spatial scales, etc. In all cases, these representations satisfy the wave equation, but each is subject to different boundary conditions and simplifying assumptions. Each is therefore relevant to particular types of subsurface investigation. Below is a table summarizing the salient characteristics of the three.

Seismic Domains	Type		Application	Data	Boundary Conditions
Body Waves	P-SV	SH	High frequency	travel times; waveforms	unbounded
Surface Waves	Rayleigh	Love	Lithosphere	dispersion; group $c(w)$ & phase $u(w)$ velocities	interfaces
Normal Modes	Spheroidal Modes	Toroidal Modes	Global	power spectra	spherical earth

Figure 1: Summary of 3 subdisciplines of seismology.

As the table suggests, the normal modes provide a framework for representing global seismic waves. Typically, these modes of free oscillation are of extremely low frequency and are therefore difficult to observe in seismograms. Only the most energetic earthquakes are capable of generating free oscillations that are readily apparent on most seismograms, and then only if the seismograms extend over several days.

### Normal Modes (Free Oscillations)

Please note that normal modes were only touched on in the 2008 semester course. For further discussion of Normal Modes in the Fourier Domain and Synthetic Seimograms, see 22 April 2005 open courseware materials.

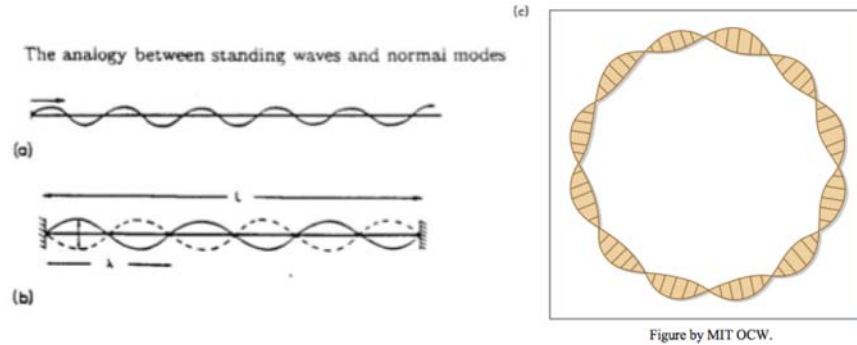


Figure 2: (b) 1D and (c) 3D analogs of free oscillation.

To understand normal modes, which describe the modes of free oscillation of a sphere, it's instructive to consider the 1D analog of a vibrating string fixed at both ends as shown in panel Figure 2b. This is useful because the 3D case (Figure 2c), similar to the 1D case, requires that standing waves wrap around and meet at a null point. The string obeys the 1D wave equation with fixed-end BCs, the general expression and solution to which are:

$$(1) \quad \frac{\delta^2 u}{\delta x^2} = \frac{1}{c^2} \frac{\delta^2 u}{\delta t^2}$$

$$\therefore u(x, t) = Ae^{i(t+x/c)} + Be^{i(t-x/c)} + Ce^{i(t+x/c)} + De^{i(t-x/c)}$$

The boundary conditions require that  $u(0, t) = u(L, t)$ , which implies that  $A = -B$  and  $C = -D$ . Hence:

$$(2) \quad 2i(Ae^{it} - Ce^{-it}) \sin\left(\frac{\omega L}{c}\right) = 0 \Rightarrow \sin\left(\frac{\omega L}{c}\right) = 0$$

$$\Rightarrow \frac{\omega L}{c} = (n+1)\pi, n = 0, 1, 2, \dots, \infty$$

So there are infinitely many discrete frequencies,  $\omega_n$ , that satisfy (1), and these are called eigenfrequencies.

Figure 3 depicts several modes or eigenfrequencies that satisfy (2).  $n=0$  corresponds to the fundamental mode and all  $n \geq 1$  correspond to higher modes (overtones).

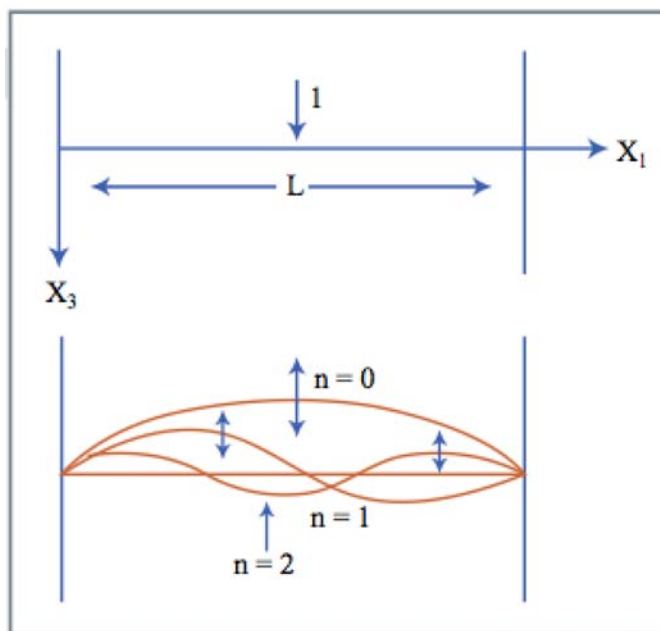


Figure by MIT OCW.

Figure 3: Modes ( $n$ ).

Aside: We have already seen  $\omega - k$  plots for surface and body waves and have learned how to interpret and manipulate them. Normal modes are also frequently graphically depicted using  $\omega - l$  plots, where  $\omega$  has the normal meaning and where  $l$  is the characteristic length or *angular order*. But note  $k = \frac{2\pi}{\lambda}$  and  $l = \frac{2\pi R}{\lambda}$  where  $\lambda$  is the wavelength; so the angular order is like a wave number!  $l$  also denotes the spherical harmonic degree.

**Normal Mode Nomenclature** The wave equation, subject to spherical boundary conditions, gives rise to the so-called spherical harmonics:

$$\ddot{u} = c^2 \nabla^2 u \rightarrow (\text{spherical boundary conditions}) \rightarrow \text{Spherical harmonics.}$$

For example, the gravitational potential can be expressed in spherical harmonics by:

$$U_{\text{grav}} = \sum_{l=0}^{\infty} \sum_{m=0}^l \left(\frac{1}{r}\right)^l \left\{ \underbrace{A_n^m \cos m\varphi + B_n^m \sin m\varphi}_{\text{Longitude-dependent}} \right\} \underbrace{P_l^m(\cos\theta)}_{\substack{\text{Legendre Polynomial} \\ \text{Latitude-dependent}}} .$$

The above equation displays a  $2l + 1$  degeneracy. That is, for each  $l$  there exist  $2l + 1$  modes (solutions). For example, for  $l = 0$  there is only one mode; for  $l = 1$ , there are three modes corresponding to  $A_1^0, A_1^1, B_1^1$ .  $n$  indicates the number of nodes along the radius of the Earth (also called the overtone number), and  $l$  is the angular order, which indicates the number of nodal planes on the surface (see Figures 4 and 5).

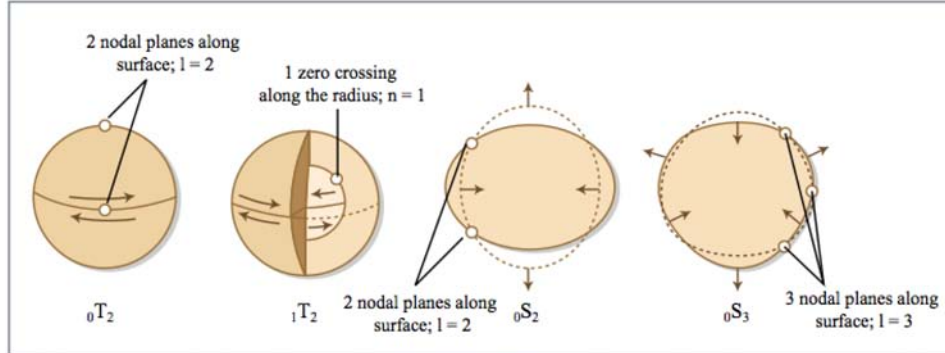


Figure by MIT OCW.

Figure 4: Spheroidal and Toroidal Motion Diagrams

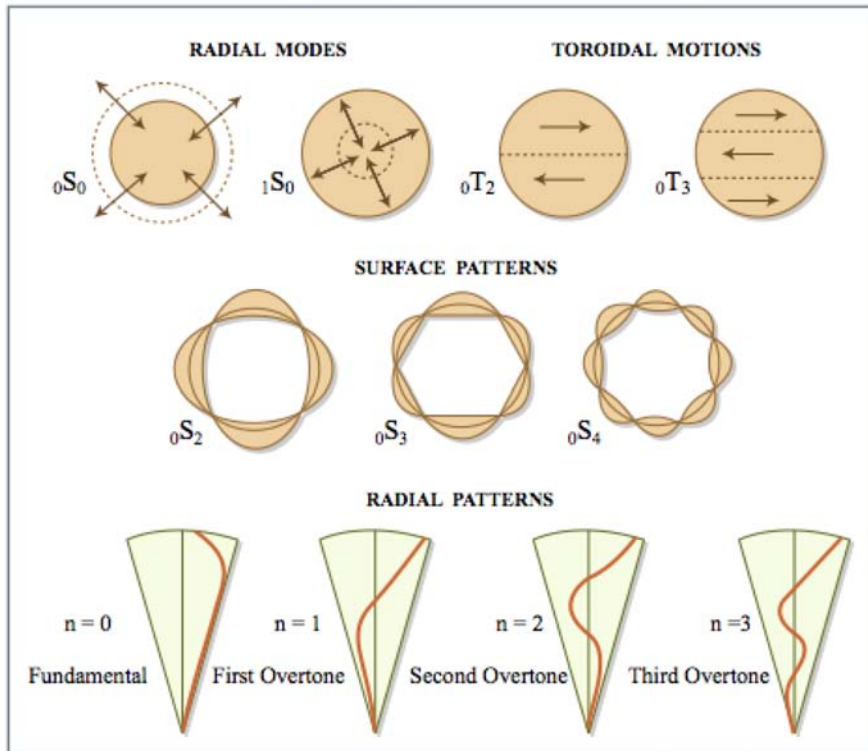


Figure 5: Surface and Radial patterns of Normal Modes

Spheroidal modes ( P-SV; changes in volume) are denoted by  ${}_nS_l$  and are sensitive to compressional and shear wavespeed as well as density. Toroidal modes ( SH; rotation or shear; no change in volume) are denoted by  ${}_nT_l$  and are sensitive only to shear wavespeed. There are more spheroidal than toroidal modes.

There are a number of modes that have been given special names. One such mode is  ${}_0S_0$  (see Figure 5, bottom left), which is called the breathing mode because the entire spherical volume periodically expands and contracts. It was first observed by normal mode seismologists after the 2004 Sumatra earthquake. Another is  ${}_0S_2$ (see Figure 5, top left), which is called the football mode because the extrema of this free oscillation are shaped like an American football (also because a European football displays this oscillation when it is kicked). Two modes that do not exist naturally are the  ${}_0S_1$  and  ${}_0T_1$  modes. The  ${}_0S_1$  spheroidal mode cannot exist because it requires the displacement of the center of gravity, which cannot happen. The  ${}_0T_1$  toroidal mode cannot exist because it requires the entire sphere to twist back and forth, which contradicts the conservation of angular momentum for a rotating sphere.

In 3D spherical harmonics, each spherical harmonic or normal mode of the Earth can be treated as a basis function. Any wave on Earth can be represented as a weighted sum of normal modes or spherical harmonics, as heuristically expressed in the  $x - t$  domain. Therefore, if we take the normal mode power spectrum of a seismogram we shall see spectral peaks corresponding to the frequency of these modes. But, because of the  $2l + 1$  degeneracy described above, there are multiple modes of free oscillation for each  $l$ . Each of these modes should oscillate at the same frequency. However, because of the Earth's rotation, the  $2l + 1$  modes will not be observed to be oscillating at precisely the same frequency. Hence, while we should observe a sharp spike in the power spectrum for each  $l$  (this is called a *singlet*), we often observed a broadened or smeared spike around the expected frequency of the normal mode (this is called a *multiplet*). These ideas are illustrated in Figure 7.

In fact, a multiplet can sometimes become so spread out that the power spectrum for a particular mode appears multimodal. This is aptly termed *normal mode splitting*. It is currently hypothesized that normal mode splitting is the effect of anisotropy, not rotational frequency modulation.

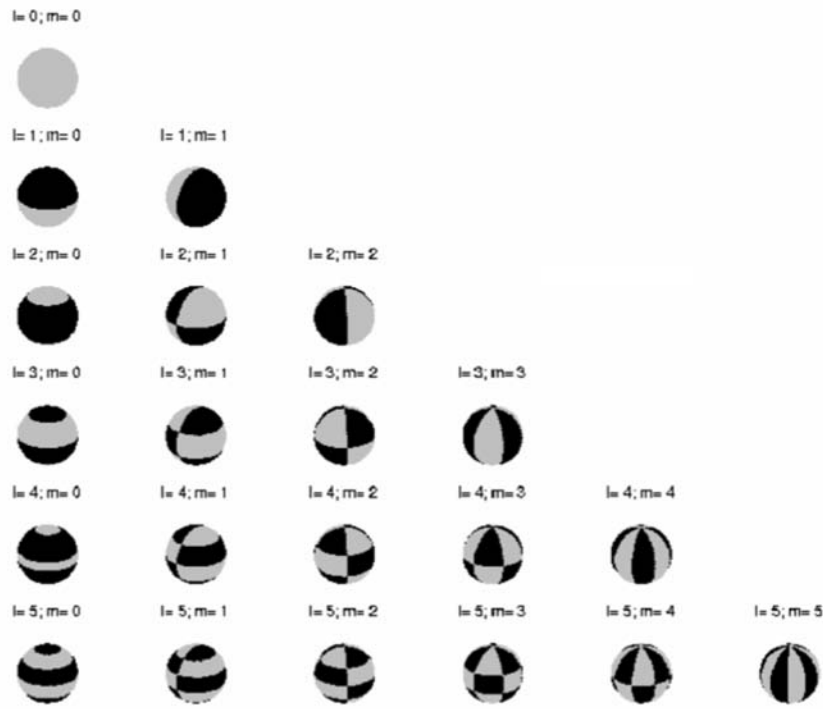


Figure 6: Some spherical harmonics

### Normal Mode Seismology: Density

Density is very poorly constrained by body and surface waves. Free Oscillations are constrained by gravity and tell us about density structure:  
 $\rho \ddot{u} = \nabla \cdot \sigma + f_{gravity}$

### Reflectivity Method

A Propagator matrix describes a propagating wave field through multiple layers. A displacement field is connected to the adjoining layers by boundary conditions. The matrix system allows a stepwise relationship between layers called the Reflectivity Method.

Receiver functions are related to studies with interfaces and body waves and are used for inverse scattering problem. (See Chevrot and van der Hilst, EPSL 2000.)



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### Anisotropy

So far we have assumed isotropy, i.e. wavespeeds do not depend on the direction of wave propagation. We solved the wave equation assuming plane waves:

$$p = \frac{\sin i}{\alpha} = \frac{\sin j}{\beta} \text{ where } v_p = \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}} \text{ and } v_s = \beta = \sqrt{\frac{\mu}{\rho}}.$$

Snell's Law and Generalized Hook's law held only for isotropic medium.

Anisotropy, however, cannot be ignored as it is the focus of increasing research in seismology and is inherent in our environment from stratification, sedimentation, anisotropic minerals, etc.

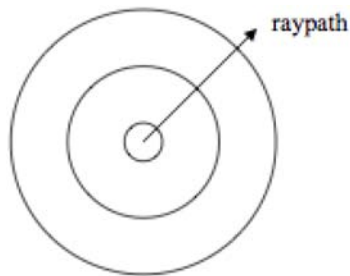


Figure 7: In an isotropic medium, wavefronts are concentric circles with radius dependent on the velocity ( $v_p$  or  $v_s$ ) as shown above. The raypaths are perpendicular to wavefronts and the slowness vector is perpendicular to the wavefront. The energy goes with group velocity and group velocity is also perpendicular to wavefronts.

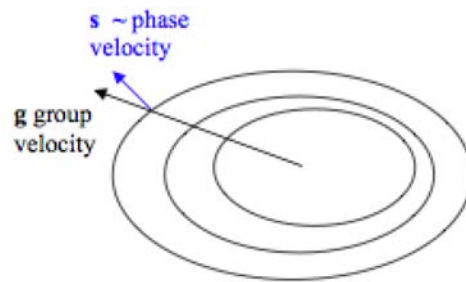


Figure 8: In an anisotropic media the wavefronts are distorted. The ray-paths are not perpendicular to the wavefronts, therefore the direction of the group velocity is not the same as the direction of the phase velocity. In a homogeneous media there are still three solutions but now they are called quasi P (q P), quasi SH(q SH) and quasi Sv (q Sv).

Now we look at some basic theory to get insights into the problem and concepts. A full treatment is beyond the scope of this class.

We have used the Generalized Hookes Law:

$$\begin{aligned}\tau_{ij} &= c_{ijkl}\epsilon_{kl} \\ \tau_{ij} &= \tau_{ji} \rightarrow c_{ijkl} = c_{jikl} \\ \epsilon_{kl} &= \epsilon_{lk} \rightarrow c_{ijkl} = c_{jilk}\end{aligned}$$

And from thermodynamics:

$$c_{ijkl} = c_{klij}$$

These relationships reduce the number of independent elements from 81 to 21 elements.

In the **isotropic case**, Generalized Hooke's Law simplifies to:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Where only the two independent parameters are the Lam constants,  $\lambda$  and  $\mu$ .

$$\begin{aligned}c_{1111} &= c_{2222} = c_{3333} = \lambda + 2\mu \\ c_{1122} &= c_{1133} = c_{2233} = \lambda \\ c_{1212} &= c_{1313} = c_{2323} = \mu \\ Else &= 0\end{aligned}$$

An **anisotropic medium** is a more complex system and uses symmetries such as:

Orthorhombic- (e.g. olivine) with 9 elements

Hexagonal- with 5 elements

Cubic- (e.g. MgO) with 3 elements

Anisotropy is not due to individual minerals but the whole medium or lattice. Lattice preferred orientation (LPO) is the deformation of olivine by plate motion. Olivine minerals can align along their long axes from high stress deformation. We can use studies of anisotropy as a measure of strain, but we have problems distinguishing history dependent LPO from signals of current strain.

### Anisotropic Case: Hexagonal Symmetry

Hexagonal symmetry is very useful for seismology where there is rotational symmetry around a symmetry axis (not necessarily vertical but often is).

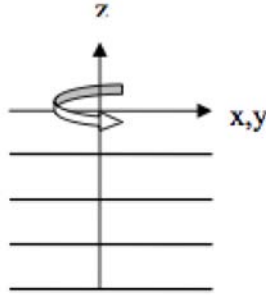


Figure 9: Anisotropic medium with hexagonal symmetry. This layered system has a vertical symmetry axis which is most useful for Earth. In the transverse direction (ie. direction perpendicular to the symmetry axis) we have isotropy. This is also known as Vertical Transverse Isotropy (VTI).

The advantages of VTI are:

1. q SH and q Sv can still be treated separately.
2. The velocities vary only with incidence angle and not with azimuth.

If the symmetry axis is horizontal you have azimuthal anisotropy or Horizontal Transverse Isotropy (HTI).

For mathematical convenience we introduce a new notation convention in order to collapse the 4th order tensor into a 2nd order tensor (ie.

$c_{ijkl} \rightarrow C_{IJ}$ ):

Let  $11 \rightarrow 1$ ;  $22 \rightarrow 2$ ;  $33 \rightarrow 3$ ;  $23 \rightarrow 4$ ;  $13 \rightarrow 5$ ;  $12 \rightarrow 6$ .

For example,  $c_{1122} = C_{12}$ .

So for isotropic media:

$$C_{ij} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

We will continue this discussion in the next class.