12.307 Weather and Climate Laboratory
Spring 2009

12.307
Convection in water (an almost-incompressible fluid)

John Marshall, Lodovica Illari and Alan Plumb

March, 2004

1 Convection in water (an almost-incompressible fluid)

1.1 Buoyancy

Objects that are lighter than water bounce back to the surface when immersed, as has been understood since the time of Archimedes. But what if the 'object' is fluid itself, as sketched in Fig.1? Let's consider the stability of such a parcel[1] in an incompressible liquid. We will suppose that density depends on temperature and not on pressure. Imagine that the parcel shaded in Fig.1 is warmer, and hence less dense than its surroundings.

If there is no motion then the fluid will be in hydrostatic balance (since ρ is uniform above) — the pressure at A_1, A and A_2 will be the same. But, because there is lighter fluid in the column above B than above either point B_1 or B_2, from integration of the hydrostatic equation, $p(z) = g \int_z^\infty \rho dz$, we see that the hydrostatic pressure at B will be less than at B_1 and B_2. Since fluid has a tendency to flow from regions of high pressure to low pressure, fluid will begin to move toward the low pressure region at B and tend to equalize the pressure along B_1BB_2; the pressure at B will tend to increase and apply an upward force to the buoyant fluid which will therefore begin to move upwards. Thus the light fluid will rise.

---

[1] A 'parcel' of fluid is imagined to have a small but finite dimension, is thermally isolated from the environment and is always at the same pressure as its immediate environment.
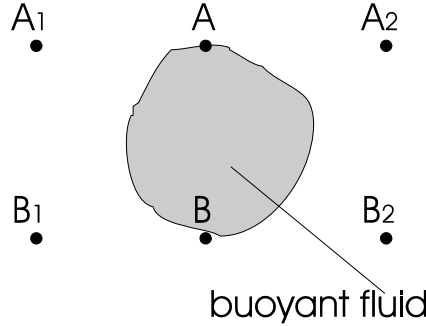
Figure 1: A parcel of buoyant fluid surrounded by resting, homogeneous, heavier fluid in hydrostatic balance. The fluid above points A_1, A and A_2 has the same density and hence, as can be deduced by consideration of hydrostatic balance, the pressures at the A points are all the same. But the pressure at B is lower than at B_1 or B_2 because the column of fluid above it is lighter. There is thus a pressure gradient force which drives fluid inwards toward B, forcing the light fluid upward.

In fact the acceleration of the parcel of fluid is not g but $g\frac{\Delta\rho}{\rho_P}$ where $\Delta\rho = (\rho_P - \rho_E)$. The factor $g\frac{\Delta\rho}{\rho_P}$ is known as the 'reduced gravity'. It is also common to speak of the buoyancy of the parcel defined by:

$$b = -g\frac{(\rho_P - \rho_E)}{\rho_P} \tag{1}$$

If $\rho_P < \rho_E$ then the parcel is *positively buoyant* and rises: if $\rho_P > \rho_E$ the parcel is *negatively buoyant* and sinks: if $\rho_P = \rho_E$ the parcel is *neutrally buoyant* and neither sinks or rises.

Let's now consider this problem in terms of the stability of the fluid 'parcel'.

1.2   Stability

Suppose we have a horizontally uniform state with temperature $T(z)$ and density $\rho(z)$. T and ρ are assumed to be related by an equation of state which tells us how the density of water depends on the temperature:

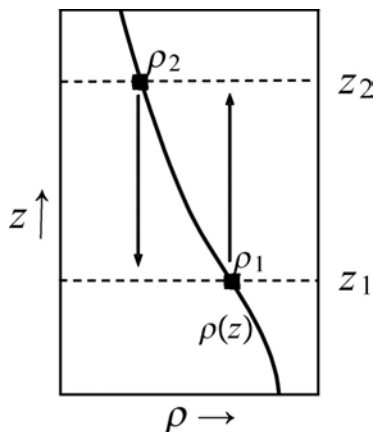$$\rho = \rho_{ref}(1 - \alpha[T - T_{ref}]) \tag{2}$$

Figure 2: We consider a fluid parcel initially located at height z_1 in an environment whose density is $\rho(z)$. It has density $\rho_1 = \rho(z_1)$, the same as its environment at height z_1. It is now displaced adiabatically a small vertical distance to $z_2 = z_1 + \delta z$ where its density is compared to that of the environment.

is a good approximation for (fresh) water in typical circumstances, where ρ_{ref} is a constant reference value of the density and α is the coefficient of thermal expansion at $T = T_{ref}$.

Again we focus attention on a single fluid parcel, initially located at height z_1. It has temperature $T_1 = T(z_1)$ and density $\rho_1 = \rho(z_1)$, the same as its environment; it is therefore *neutrally buoyant*, and thus in equilibrium. Now let us displace this fluid parcel a small vertical distance to $z_2 = z_1 + \delta z$, as shown in Fig.2.

We are going to figure out the buoyancy of the parcel when it arrives at height z_2. Now, if the displacement is done sufficiently rapidly so that the parcel does not lose or gain heat on the way, it will occur *adiabatically* and the temperature T will be conserved during the displacement. This is a reasonable assumption because the temperature of the parcel can only change by diffusion, which is a slow process compared to typical fluid movements and can be neglected here. Therefore the temperature of the perturbed parcel at z_2 will still be T_1, and so its density will still be ρ_1. The environment, however, has density

$$\rho(z_2) \simeq \rho_1 + \left(\frac{d\rho}{dz}\right)_E \delta z \ ,$$

3

where $(d\rho/dz)_E$ is the environmental density gradient. The buoyancy of the parcel just depends on the difference between its density and that of its environment; therefore it will be

$$\left.\begin{array}{c} \text{positively} \\ \text{neutrally} \\ \text{negatively} \end{array}\right\} \text{ buoyant if } \left(\frac{d\rho}{dz}\right)_E \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0 \end{array}\right. . \tag{3}$$

where positively buoyant means the parcel has a density *less* than its environment. If the parcel is positively buoyant (the situation sketched in Fig.1), it will keep on rising at an accelerated rate. Therefore an incompressible liquid is **unstable if density increases with height** (in the absence of viscous and diffusive effects). It is this instability that leads to the convective motions discussed above. Using Eq.(2) the stability condition can also be expressed in terms of temperature as:

$$\left.\begin{array}{c} \text{UNSTABLE} \\ \text{NEUTRAL} \\ \text{STABLE} \end{array}\right\} \text{ if } \left(\frac{dT}{dz}\right)_E \left\{\begin{array}{l} < 0 \\ = 0 \\ > 0 \end{array}\right. . \tag{4}$$

Note that Eq.(4) is appropriate for an incompressible fluid whose density only depends on temperature.

1.3 Energetics

Let's study our problem from yet another angle. One way is to consider energy changes, for we know that if the potential energy of a parcel can be decreased and converted into motion, then this is likely to happen.

Consider now two small parcels of incompressible fluid of equal volume at differing heights, z_1 and z_2 as sketched in Fig.2. Because the parcels are incompressible they do not expand or contract as p changes and so do not do work on, or have work done on them, by the environment. This greatly simplifies consideration of energetics. The potential energy of the initial state is:

$$PE_{initial} = g\left(\rho_1 z_1 + \rho_2 z_2\right).$$

Now let's interchange the particles. The potential energy of the state after swapping — the 'final' state — is

$$PE_{final} = g\left(\rho_1 z_2 + \rho_2 z_1\right).$$

The change in PE, ΔPE is given by:

$$\begin{aligned}
\Delta PE &= PE_{final} - PE_{initial} = -g\left(\rho_2 - \rho_1\right)\left(z_2 - z_1\right) \quad (5)\\
&\simeq -g\left(\frac{d\rho}{dz}\right)_E \left(z_2 - z_1\right)^2
\end{aligned}$$

where $\left(\frac{d\rho}{dz}\right)_E = \frac{(\rho_2 - \rho_1)}{(z_2 - z_1)}$ is the mean density gradient of the environmental state. Note that the factor $g\left(z_2 - z_1\right)^2$ is always positive and so the sign of ΔPE depends on $\left(\frac{d\rho}{dz}\right)_E$.

Hence if $\left(\frac{d\rho}{dz}\right)_E > 0$ then rearrangement leads to a decrease in ΔPE and the possibility of growth of kinetic energy of the parcels, *i.e.* a disturbance is likely to grow — the system will be unstable. But if $\left(\frac{d\rho}{dz}\right)_E < 0$ then ΔPE is positive and energy cannot be released by exchanging parcels. So we again arrive at the stability criterion, Eq.(4). This energetic approach is simple but very powerful. It should be emphasized, however, that we have only demonstrated the possibility of instability. To show that instability is a fact, one must carry out a stability analysis (we are not going to do this here) in which the details of the perturbation are worked out. However, when energetic considerations point to the possibility of convective instability, exact solutions of the governing dynamical equations almost invariably show that instability is a fact.

1.4   GFD Lab: Convection

We can study convection in the laboratory using the apparatus shown in Fig.3. A heating pad at the base of the tank triggers convection in an initially (temperature) stratified fluid. Convection carries heat from the heating pad into the body of the fluid distributing it over the convection layer, much like convection carries heat from the Earth's surface vertically.

Thermals can be seen to rise from the heating pad, entraining fluid as they rise. Parcels overshoot the level at which they become neutrally buoyant and brush the stratified layer above generating gravity waves on the inversion — see Fig.4 — before sinking back in to the convecting layer beneath. Successive thermals rise higher as the layer deepens. The net effect of convection is
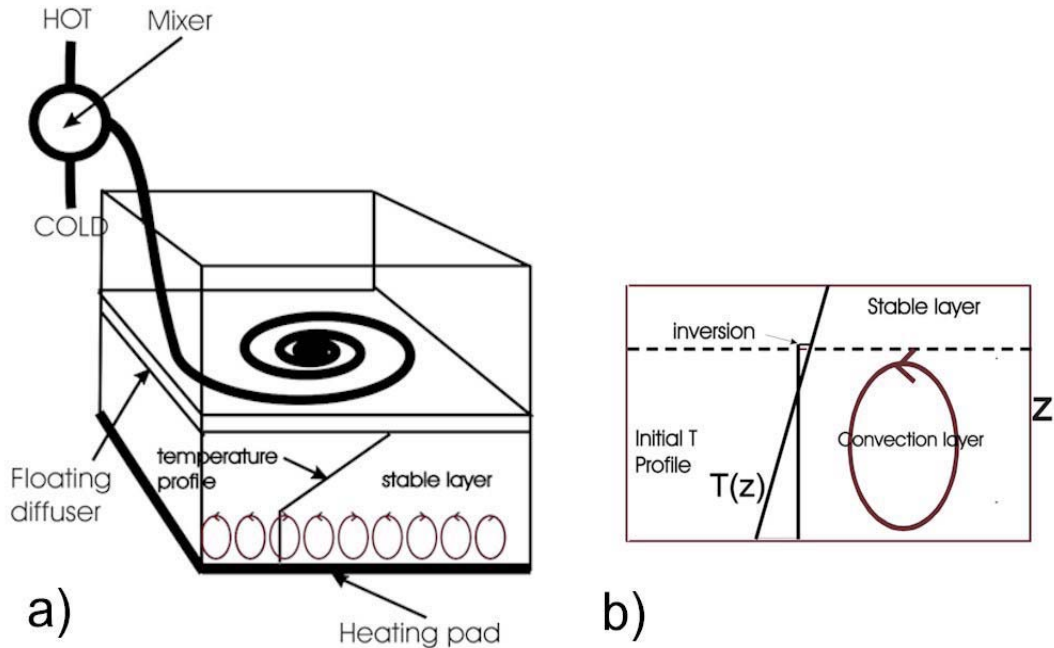
Figure 3: (a) A sketch of the laboratory apparatus used to study convection. A stable stratification is set up in a 50cm^3 tank by slowly filling it up with water whose temperature is slowly increased with time. This is done using 1) a mixer which mixes hot and cold water together and 2) a diffuser which floats on the top of the rising water and ensures that the warming water floats on the top without generating turbulence. Using the hot and cold water supply we can achieve a temperature difference of 20°C over the depth of the tank. The temperature profile is measured and recorded using thermometers attached to the side of the tank. Heating at the base is supplied by a heating pad. The motion of the fluid is made visible by sprinkling a very small amount of potassium permanganate evenly over the base of the tank after the stable stratification has been set up and just prior to turning on the heating pad. (b) Schematic of evolving convective boundary layer heated from below. The initial linear temperature profile is T_E. The convection layer is mixed by convection to a uniform temperature.. Fluid parcels overshoot in to the stable stratification above creating an inversion, Both the temperature of the convection layer and its depth slowly increase with time.
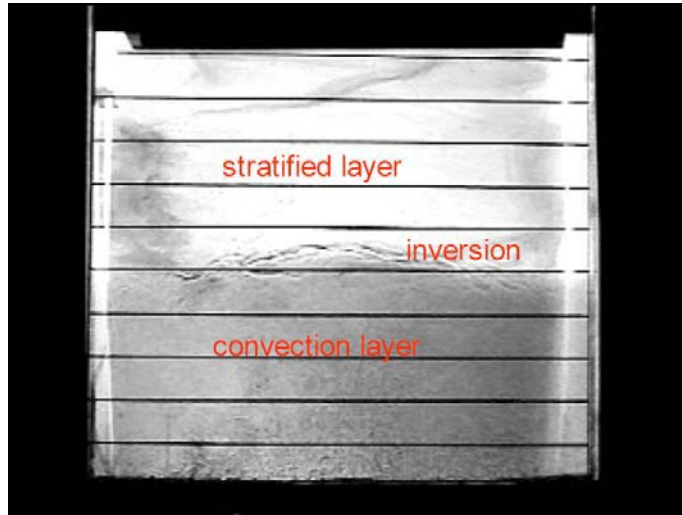
6

Figure 4: A snapshot of the convecting boundary layer in the laboratory expriment. Note the undulations on the inversion caused by convection overshooting the well mixed layer below in to the stratified layer above.

to erode the vertical stratification, returning the fluid to a state of neutral stability — in this case a state in which the temperature of the convecting layer is close to constant with near vanishing vertical gradients, as sketched in the schematic, Fig.3.

Fig.5 plots T timeseries measured by thermometers at various heights above the heating pad (see legend for details). We observe an initial temperature difference of some 18°C from top to bottom. After the heating pad is switched on, T increases with time, first for the bottom most thermometer but, subsequently, as the convecting layer deepens, for thermometers at each successive height as they begin to measure the temperature of the convecting layer. Note how by the end of the experiment that T is rising simultaneously at all heights within the convection layer. We see, then, that the convection layer is well mixed, and essentially of spatially uniform temperature. Closer inspection of the $T(t)$ curves reveals fluctuations of order $\pm\ 0.1$ °C associated with individual convective events within the fluid.

1.4.1 Law of vertical heat transport

We can make use of energetic considerations to develop a simple 'law of vertical heat transport' for the convection in our tank. We found that the change in potential energy resulting from the interchange of the two small parcels of (incompressible) fluid is given by Eq.(5). Let us now assume that the PE released in convection (as light fluid rises and dense fluid sinks) is acquired by the kinetic energy of the convective motion:

$$\frac{3}{2}\rho_{ref}w^2 = -g\left(\rho_2 - \rho_1\right)\left(z_2 - z_1\right)$$

where we have assumed that the convective motion is isotropic in the three directions of space with speed w.

Now using our equation of state for water, Eq.(2), we may simplify the above to:

$$w^2 \simeq \frac{2}{3}\alpha g \Delta z \Delta T \tag{6}$$

where ΔT is the difference in temperature between the upwelling and downwelling parcels which are exchanged over a height $\Delta z = z_2 - z_1$, and w is a typical vertical velocity.

Eq.(6) implies the following "law" of vertical heat transfer for the convection in our tank (using c_p, the specific heat of water to convert the vertical temperature flux to a heat flux which has units of Wm^{-2}):

$$\mathcal{H} = \rho_{ref}c_p\overline{w\Delta T}^{time} = \rho_{ref}c_p\left(\alpha g \Delta z\right)^{\frac{1}{2}}\Delta T^{\frac{3}{2}} \tag{7}$$

where $\overline{()}^{time}$ is a time average over many convective events.

In the convection experiment shown in Fig.4, the heating pad supplied energy at around $\mathcal{H} = 4000Wm^{-2}$. If the convection penetrates over a vertical scale $\Delta z = 0.2m$, then Eq.(7) implies $\Delta T \simeq 0.1K$ if $\alpha = 2\times10^{-4}K^{-1}$ and $c_p = 4000JKg^{-1}K^{-1}$. Eq.(6) then implies a parcel speed of $\simeq 0.5cm$ s^{-1}. This is not untypical of what is observed in the experiment.

We see that in order to transfer heat away from the pad vertically through the fluid, vigorous convection ensues. Even though the temperature variations within the convection layer are small (only $\simeq 0.1K$) they are sufficient to accomplish the transfer. Moreover, we see that the convection layer is very well mixed, as can be seen in the $T(t)$ observations in Fig.5 and sketched in Fig.3.
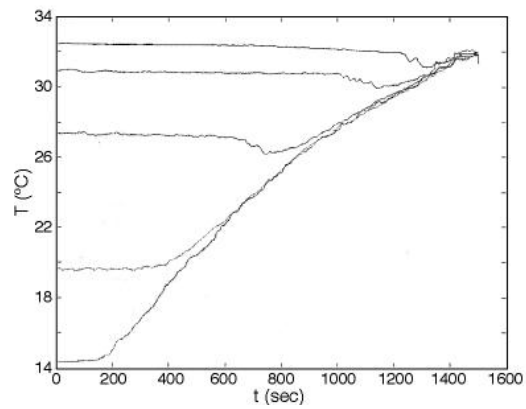
Figure 5: Temperature timeseries measured by 5 thermometers spanning the depth of the fluid at equal intervals. The lowest thermometer is close to the heating pad. We see that the ambient fluid initally has a roughly constant stratification, somewhat higher near the top than in the body of the fluid. The heating pad was switched on at $t = 170s$. Note how all the readings converge on to one line as the well mixed convection layer deepens over time.