12.215 Modern Navigation

Thomas Herring

Review of Wednesday Class

- Definition of heights
  - Ellipsoidal height (geometric)
  - Orthometric height (potential field based)
- Shape of equipotential surface: Geoid for Earth
- Methods for determining heights

Today's Class

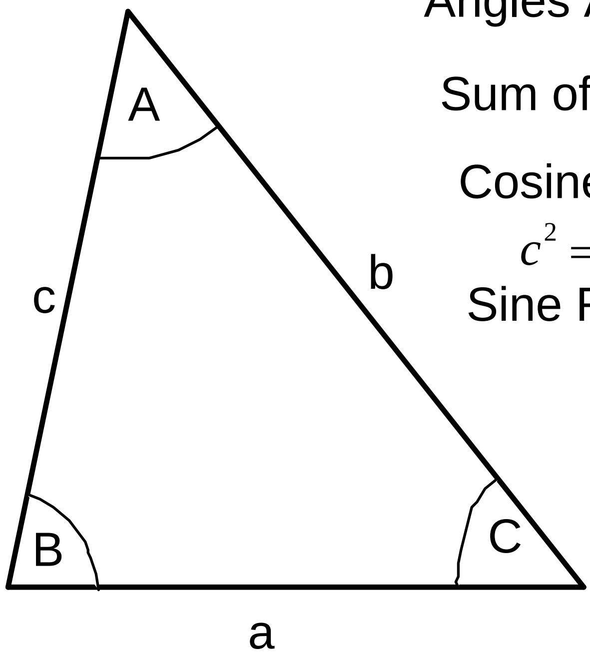
- Spherical Trigonometry
  - Review plane trigonometry
  - Concepts in Spherical Trigonometry
    - Distance measures
    - Azimuths and bearings
  - Basic formulas:
    - Cosine rule
    - Sine rule
- http://mathworld.wolfram.com/SphericalTrigonometry.html is a good explanatory site

Spherical Trigonometry

- As the name implies, this is the style of trigonometry used to calculate angles and distances on a sphere
- The form of the equations is similar to plane trigonometry but there are some complications. Specifically, in spherical triangles, the angles do not add to 180^o
- "Distances" are also angles but can be converted to distance units by multiplying the angles (in radians) by the radius of the sphere.
- For small sized triangles, the spherical trigonometry formulas reduce to the plane form.

Review of plane trigonometry

- Although there are many plane trigonometry formulas, almost all quantities can be computed from two formulas: The cosine rule and sine rules.



Angles A, B and C; Sides a, b and c

Sum of angles A+B+C=180

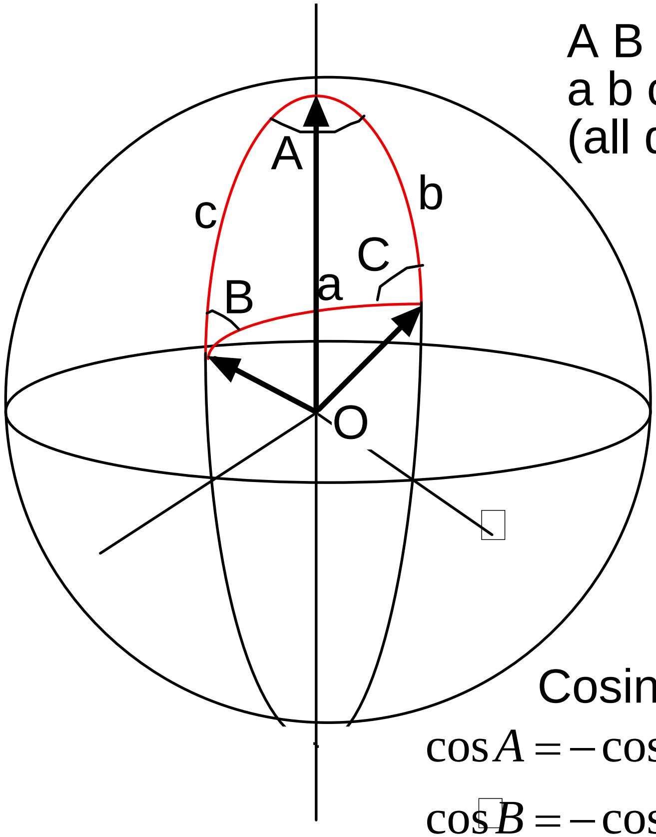
Cosine Rule:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Sine Rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Basic Rules
(discussed in following slides)



A B C are angles
a b c are sides
(all quanties are angles)

Sine Rule

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

Cosine Rule sides

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos c \cos a + \sin c \sin a \cos B$$
$$\cos c = \cos b \cos a + \sin a \sin b \cos C$$

Cosine Rule angles

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$
$$\cos B = -\cos A \cos C + \sin A \sin C \cos b$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c$$

Spherical Trigonometry Interpretation

- Interpretation of sides:
  - The spherical triangle is formed on a sphere of unit radius.
  - The vertices of the triangles are formed by 3 unit vectors (OA, OB, OC).
  - Each pair of vectors forms a plane. The intersection of a plane with a sphere is a circle.
    - If the plane contain the center of the sphere (O), it is called a great circle
    - If center not contained called a small circle (e.g., a line of latitude except the equator which is a great circle)
  - The side of the spherical triangle are great circles between the vertices. The spherical trigonometry formulas are only valid for triangles formed with great circles.

Interpretation

- Interpretation of sides (continued):
  - Arc distances along the great circle sides are the side angle (in radians) by the radius of the sphere. The side angles are the angles between the vectors.
- Interpretation of angles
  - The angles of the spherical triangles are the dihedral angles between the planes formed by the vectors to the vertices.
  - One example of angles is the longitude difference between points B and C if A is the North Pole.

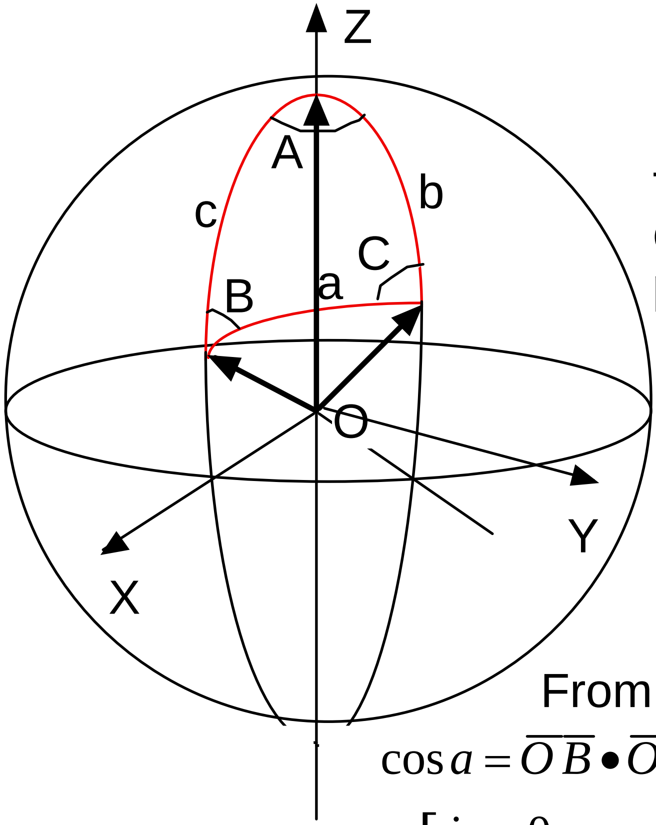
Interpretation

- In navigation applications the angles and sides of spherical triangles have specific meanings.
- Sides: When multiplied by the radius of the Earth, are the great circle distances between the points. On a sphere, this is the short distance between two points and is called a geodesic. When one point is the North pole, the two sides originating from that point are the co-latitudes of the other two points
- Angles: When one of the points is the North pole, the angles at the other two points are the azimuth or bearing to the other point.

Derivation of Cosine rule

- The spherical trigonometry cosine rule can be derived form the dot product rule of vectors fairly easily. The sine rule can be also derived this way but it is more difficult.
- On the next page we show the derivation by carefully selecting the coordinate axes for expressing the vector.
- (Although we show A in the figures as at the North pole this does not need to be case. However, in many navigation application one point of a spherical triangle is the North pole.)

Derivation of cosine rule



Taking OA as the Z-axis, and OB projected into the plane perpendicular to OA as the X axis

Vector OB has components:
[sin c, 0, cos c]

Vector OC has components:
[sin b cos A, sin b sin A, cos b]

From dot product rule:

$$\cos a = \overline{O}\,\overline{B} \bullet \overline{O}\,\overline{C}$$

$$= [\sin c, 0, \cos c] \bullet [\sin\ b\ \cos\ A,\ \sin\ b\ \sin\ A,\ \cos\ b]$$

$$= \sin c \sin b \cos A + \cos c \cos b$$

Area of spherical triangles

- The area of a spherical triangle is related to the sum of the angles in the triangles (always >180^o)
- The amount the angles in a spherical triangle exceed 180^o or π radians is called the spherical excess and for a unit radius sphere is the area of the spherical triangle
  - e.g. A spherical triangle can have all angles equal 90^o and so the spherical excess is $\pi/2$. Such a triangle covers 1/8 the area of sphere. Since the area of a unit sphere is 4π ster-radians, the excess equals the area (consider the triangle formed by two points on the equator, separated by 90^o of longitude).

Typical uses of Spherical Trigonometry

- Spherical trigonometry is used for most calculations in navigation and astronomy. For the most accurate navigation and map projection calculation, ellipsoidal forms of the equations are used but this equations are much more complex and often not closed formed.
- In navigation, one of the vertices is usually the pole and the sides b and c are colatitudes.
- The distance between points B and C can be computed knowing the latitude and longitude of each point (Angle A is difference in longitude) using the cosine rule. Quadrant ambiguity in the $\cos^{-1}$ is not a problem because the shortest distance between the points is less than 180^{o}
- The bearing between the points is computed from the sine rule once the distance is known.

Azimuth or Bearing calculation

- In the azimuth or bearing calculation, quadrant ambiguity is a problem since the $\sin^{-1}$ will return two possible angles (that yield the same value for sine).
- To resolve the quadrant ambiguity: rewrite the cosine rule for sides (since all sides are now known) in the form:
  $\cos B = (\cos b - \cos c \cos a)/(\sin c \sin a)$
- Then use the two-argument $\tan^{-1}$ to compute the angle and correct quadrant.
- Note: That the aziumuth to travel back to point is not simply 180^o+forward azimuth. To stay on a great circle path, the azimuth or bearing of travel has to change along the parth.

Homework

- Homework 1 is available on the 12.215 web page
  12.215_HW01
- Due Wednesday October 4, 2006
- Solutions to the homework can be submitted on paper or by email as a PDF or Word document.

Summary

- In this class, we
  - Reviewed plane trigonometry
  - Introduced concepts in Spherical Trigonometry
    - Distance measures
    - Azimuths and bearings
  - Developed the basic formulas:
    - Cosine rule
    - Sine rule
  - Discuss some ways these formulas are used