12.005 Lecture Notes 25

Navier-Stokes Equation – dimensional form

$$-\frac{\partial p}{\partial x_i}+\eta\frac{\partial^2 v_i}{\partial x_j \partial x_j}+f_i=\rho\frac{Dv_i}{Dt}$$

Assume:

Characteristic velocity v_0

Characteristic length L

Characteristic stress $\eta v_0 / L$

Characteristic time v_0 / L

Choose non-dimensional variables

$$v' = v / v_0 \quad x' = x / L \quad p' = \frac{L}{\eta v_0} p \quad t' = \frac{v_0}{L} t$$

or

$$v = v' v_0 \quad x = x' L \quad p = \frac{\eta v_0}{L} p' \quad t' = \frac{L}{v_0} t'$$

$$\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \quad f' = \frac{L^2}{\eta v_0} f \quad \frac{\partial}{\partial t} = \frac{v_0}{L}\frac{\partial}{\partial t'}$$

Substitute into Navier-Stokes equation

$$-\frac{1}{L}\frac{\eta v_0}{L}\frac{\partial p'}{\partial x_i'}+\frac{1}{L^2}v_0\eta\frac{\partial^2 v_i'}{\partial x_j'\partial x_j'}+\frac{1}{L^2}v_0\eta f_i' = \rho v_0\frac{v_0}{L}\frac{Dv_i'}{Dt'}$$

or

$$-\frac{\partial p'}{\partial x_i'}+\frac{\partial^2 v_i'}{\partial x_j'\partial x_j'}+f_i' = \frac{\rho v_0 L}{\eta}\frac{Dv_i'}{Dt'}$$

where $\frac{\rho v_0 L}{\eta} = \mathrm{Re}$

$\Rightarrow$ Re gives importance of inertial terms relative to viscous terms.

$\text{Re} \ll 1$ viscous forces $\approx$ balance

acceleration negligible

$\text{Re} \gg 1$ inertia dominates

Note: in dimensionless form, Re is the only parameter in the Navier-Stokes equation.

$\Rightarrow$ for given geometry (boundary conditions) ALL equivalent

(non-dimensional) problems at same Re give same result!

Examples:

1. Low Reynolds number flow past a cylinder.

   $\text{Re} \ll 1$ Symmetry, like in the sphere problem.

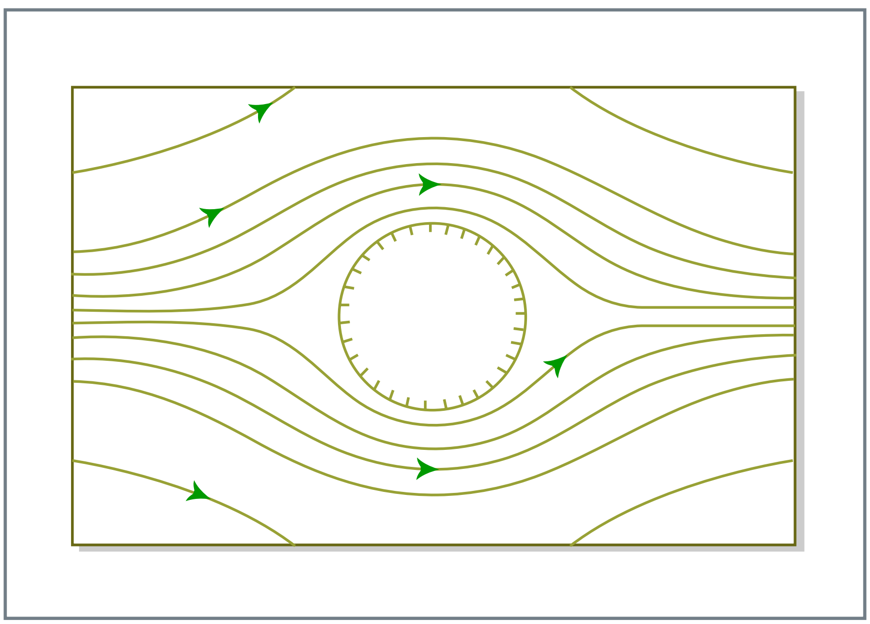

Figure 25.1

Figure by MIT OCW.

2. $\text{Re} = 10 \quad \frac{\partial v_i}{\partial t} = 0$ (steady) $\quad v_j \frac{\partial v_i}{\partial x_j} \neq 0$

Asymmetry; eddies in wade.

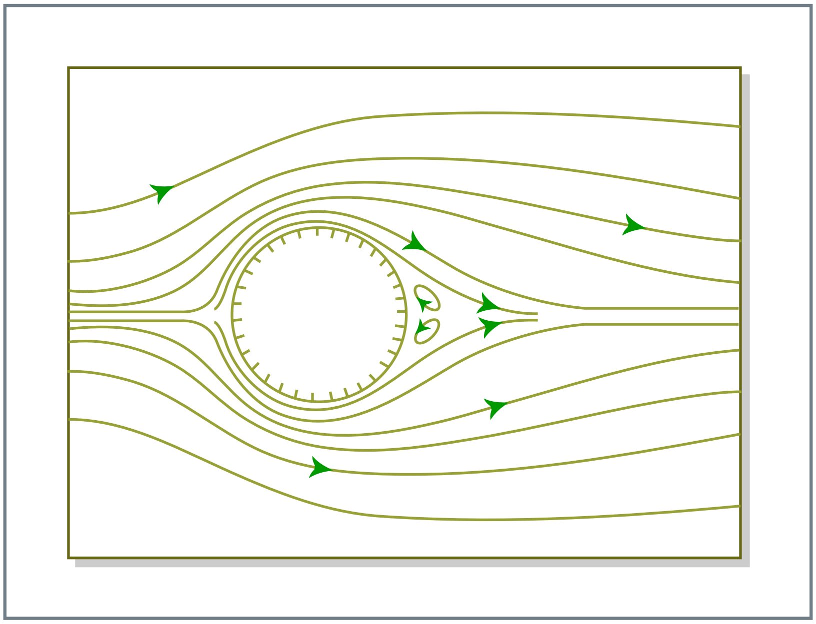

Figure 25.2

Figure by MIT OCW.

The figure below is experimental.

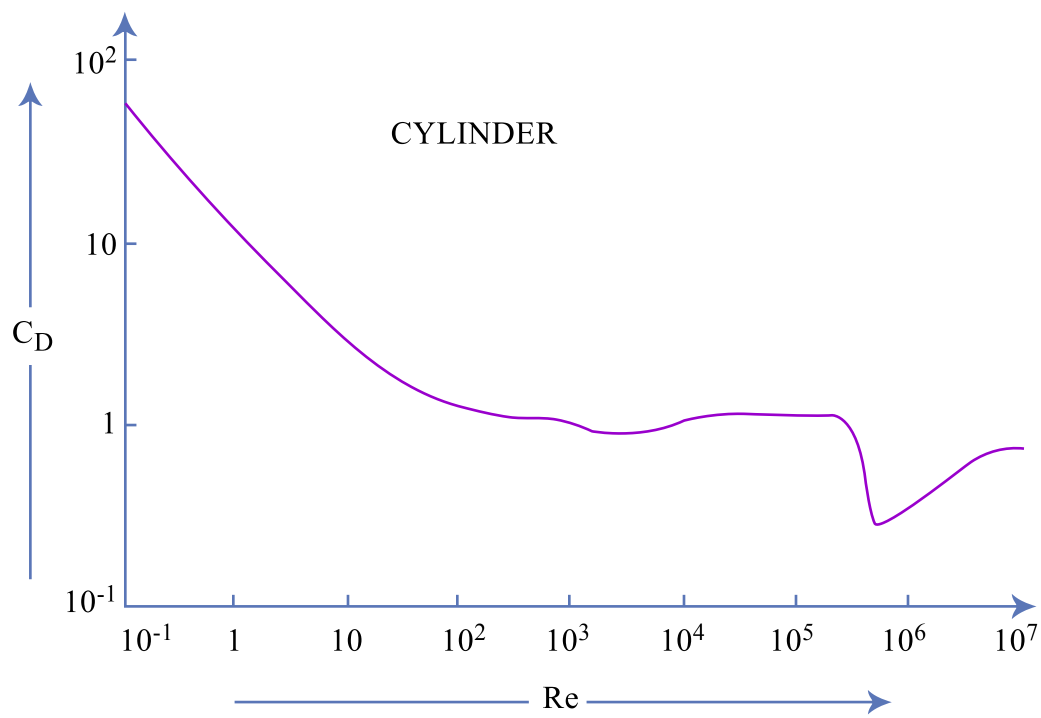

Figure 25.3. Variation of drag coefficient with Reynolds number for circular cylinder.

Figure by MIT OCW.

$Re \lesssim 1$	$C_D \sim 1 / Re$	$D \sim V$
$10^2 \lesssim Re \lesssim 3 \cdot 10^5$	$C_D \sim$ const.	$D \sim V^2$
$Re \sim 3 \cdot 10^5$	big drop in C_D!	

Recall

Earth's mantle: $Re \sim 10^{-19}$

canoe: $Re \sim 2 \cdot 10^5$

Summary:

For low Re, inertia not important

Navier-Stokes equation linear

"simple" results (analytic theory)

For high Re, inertia important

Navier-Stokes equation nonlinear

time dependent

complicated – experimental approach $\rightarrow$ empirical