4,1 Notation and key definitions

4.1.1 Deformation

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\underline{\alpha} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$\underline{\beta} = \begin{bmatrix} \cos\psi & \sin\psi \end{bmatrix}$$
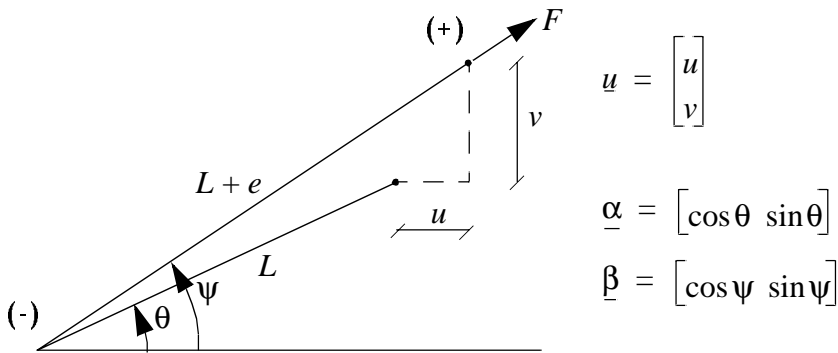
$$(L+e)^2 = (L\sin\theta + v)^2 + (L\cos\theta + u)^2$$

$$(L^2 + 2Le + e^2) = L^2 + 2eL\left(1 + \frac{e}{2L}\right) \approx L^2 + 2eL \text{ Assuming } e \ll L$$

Then

$$L^2 + 2eL \approx L^2 s^2 + 2Lsv + v^2 + L^2 c^2 + 2Lcu + u^2$$

where $s = \sin\theta$ and $c = \cos\theta$

Combine terms

$$L^2(s^2 + c^2) + 2Lsv + v^2 + 2Lcu + u^2 \approx L^2 + 2eL$$

$$e = cu + sv + \frac{1}{2L}(u^2 + v^2)$$

$$\boxed{e = \underline{\alpha}\,\underline{u} + \frac{1}{2L}\underline{u}^T\underline{u}}$$

$$\underline{\beta} = \begin{bmatrix} \cos\psi & \sin\psi \end{bmatrix}$$

$$\cos\psi = \frac{L\cos\theta + u}{L + e} = \frac{\cos\theta + u/L}{1 + e/L} \approx \cos\theta + \frac{u}{L}$$

$$\cos\psi = \frac{L\sin\theta + v}{L + e} = \frac{\sin\theta + v/L}{1 + e/L} \approx \sin\theta + \frac{v}{L}$$

Then

$$\underline{\beta} = \begin{bmatrix} \cos\theta + \dfrac{u}{L} & \sin\theta + \dfrac{v}{L} \end{bmatrix}$$

$$\boxed{\underline{\beta} = \underline{\alpha} + \frac{1}{L}\underline{u}^T}$$

4.1.2 Force

$$F = \frac{AE}{L}e$$

$\vec{F}$ has direction defined by ψ
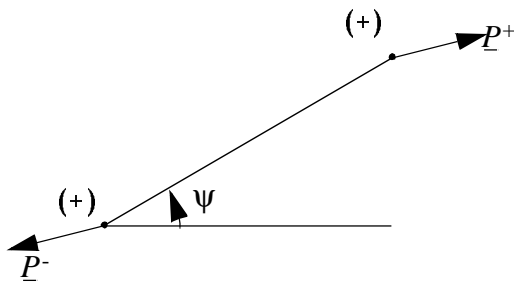
Define

$\underline{P}$ = End action matrix

$$\underline{P} = \begin{bmatrix} F\cos\psi \\ F\sin\psi \end{bmatrix} = F\underline{\beta}^T$$

At positive end $\qquad \underline{P}\big|_{+end} = +P$

At negative end $\qquad \underline{P}\big|_{-end} = -P$

4.2 Incremental Equations

4.2.1 Deformation

Consider an incremental end displacement that changes $\underline{u}$ to $\underline{u} + \Delta\underline{u}$. Find the corresponding change in elongation from e to $e + \Delta e$.

$$e = \underline{\alpha}\underline{u} + \frac{1}{2L}\underline{u}^T\underline{u}$$

$$(e + \Delta e) = \underline{\alpha}(\underline{u} + \Delta\underline{u}) + \frac{1}{2L}(\underline{u} + \Delta\underline{u})^T(\underline{u} + \Delta\underline{u})$$

$$(e + \Delta e) = \underline{\alpha}\underline{u} + \underline{\alpha}\Delta\underline{u} + \frac{1}{2L}(\underline{u}^T\underline{u} + \underline{u}^T\Delta\underline{u} + \Delta\underline{u}^T\underline{u} + \Delta\underline{u}^T\Delta\underline{u})$$

$$\Delta e = (e + \Delta e) - e = \underline{\alpha}\Delta\underline{u} + \frac{1}{L}\underline{u}^T\Delta\underline{u} + \frac{1}{2L}\Delta\underline{u}^T\Delta\underline{u}$$

$$\Delta e = de + \frac{1}{2}d^2e$$

$$de = \underline{\alpha}\Delta\underline{u} + \frac{1}{L}\underline{u}^T\Delta\underline{u} = \underline{\beta}\Delta\underline{u}$$
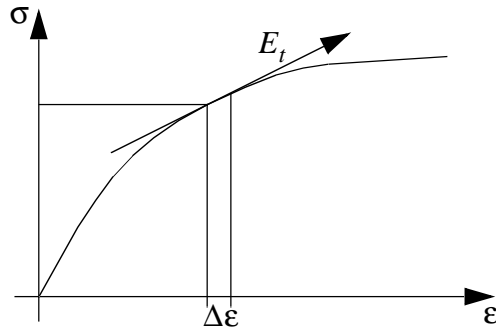
$$d^2e = \frac{1}{L}\Delta\underline{u}^T\Delta\underline{u}$$

If $\Delta\underline{u}$ is small in comparison to $\underline{u}$, it is reasonable to take $\Delta e \approx de$

Then $\qquad \boxed{\Delta e \approx \underline{\beta}\Delta\underline{u}}$

4.2.2 Force

Non Linear Stress-Strain
$$\sigma = \sigma(\varepsilon)$$

$$\sigma + \Delta\sigma = \sigma(\varepsilon + \Delta\ )$$

From a Taylor Series expansion

$$\Delta\sigma = \frac{\partial\sigma}{\partial\varepsilon}\Delta\varepsilon + \frac{1}{2}\frac{\partial^2\sigma}{\partial\varepsilon^2}\Delta\varepsilon^2 + \ldots$$

Note $\dfrac{\partial\sigma}{\partial\varepsilon} = E_t$

We want the change in stress due to the change in displacement

$$\varepsilon = \frac{1}{L}e$$

$$\Delta\varepsilon = \frac{1}{L}\Delta e = \frac{1}{L}\left(de + \frac{1}{2}d^2e\right)$$

So

$$\Delta\sigma = \frac{\partial\sigma}{\partial\varepsilon}\frac{\Delta e}{L} + \left(\frac{1}{2}\frac{\partial^2\sigma}{\partial\varepsilon^2}\Delta\varepsilon^2\right) + \ldots$$

$$\Delta\sigma = d\sigma + \frac{1}{2}d^2\sigma + \ldots$$

$$d\sigma = \frac{\partial\sigma}{\partial\varepsilon}\frac{\Delta e}{L} = \frac{E_t}{L}de = \frac{E_t}{L}\underline{\beta}\Delta\underline{u}$$

$$d^2\sigma = \frac{\partial\sigma}{\partial\varepsilon}\frac{d^2e}{L} + \frac{\partial^2\sigma}{\partial\varepsilon^2}\left(\frac{\Delta e}{L}\right)^2 \sim \frac{\partial\sigma}{\partial\varepsilon}\frac{d^2e}{L} + \frac{\partial^2\sigma}{\partial\varepsilon^2}(de)^2$$

The incremental forces are

$$F = A\sigma$$

$$\Delta F = A\Delta\sigma = Ad\sigma + \frac{A}{2}d^2\sigma$$

Set

$$dF = Ad\sigma = \frac{AE_t}{L}de = \frac{AE_t}{L}\underline{\beta}\Delta\underline{u}$$

$$d^2F = \frac{A}{2}d^2\sigma$$

$$\frac{AE_t}{L} = k_t$$

Then

$$\boxed{dF = k_t de = k_t\underline{\beta}\Delta\underline{u}}$$

4.2.3 End Actions

$$\underline{P} = F\underline{\beta}^T$$

Consider the change in $\underline{u}$ to $\underline{u} + \Delta\underline{u}$ and get the corresponding

$$(\underline{P} + \Delta\underline{P}) = (F + \Delta F)(\underline{\beta}^T + \Delta\underline{\beta}^T)$$

$$(\underline{P} + \Delta\underline{P}) = F\underline{\beta}^T + F\Delta\underline{\beta}^T + \Delta F\underline{\beta}^T + \Delta F\Delta\underline{\beta}^T$$

$$\Delta\underline{P} = (\underline{P} + \Delta\underline{P}) - \underline{P} = F\Delta\underline{\beta}^T + \Delta F\Delta\underline{\beta}^T$$

$$\Delta\underline{P} = d\underline{P} + \frac{1}{2}d^2\underline{P} + \dots$$

$$d\underline{P} = Fd\underline{\beta}^T + dF\underline{\beta}^T$$

$$\underline{\beta}(\underline{u} + \Delta\underline{u}) = \underline{\beta} + \Delta\underline{\beta} = \underline{\beta} + \frac{\partial\underline{\beta}}{\partial\underline{u}}\Delta\underline{u}^T + \frac{1}{2}\frac{\partial^2\underline{\beta}}{\partial\underline{u}^2}\Delta\underline{u}^T\Delta\underline{u}$$

$$d\underline{\beta} = \frac{\partial\underline{\beta}}{\partial u}\Delta\underline{u}^T = \left\{\frac{\partial}{\partial u}\left(\underline{\alpha} + \frac{1}{L}\underline{u}^T\right)\right\}\Delta\underline{u}^T$$

$$d\underline{\beta} = \frac{1}{L}\Delta\underline{u}^T \qquad d\underline{\beta}^T = \frac{1}{L}\Delta\underline{u}$$

So

$$\boxed{d\underline{P} = k_T\underline{\beta}^T\underline{\beta}\Delta\underline{u} + \frac{F}{L}\Delta\underline{u}}$$

Can write as

$$d\underline{P} = \underline{k}_T\Delta\underline{u}$$

where

$\underline{k}_T$ = Tangent member stiffness matrix

$$\underline{k}_T = k_T\underline{\beta}^T\underline{\beta} + \frac{F}{L}\underline{I}$$

$$\underline{k}_T = \left(\frac{AE_T}{L}\right)\underline{\beta}^T\underline{\beta} + \frac{F}{L}\underline{I}$$

where

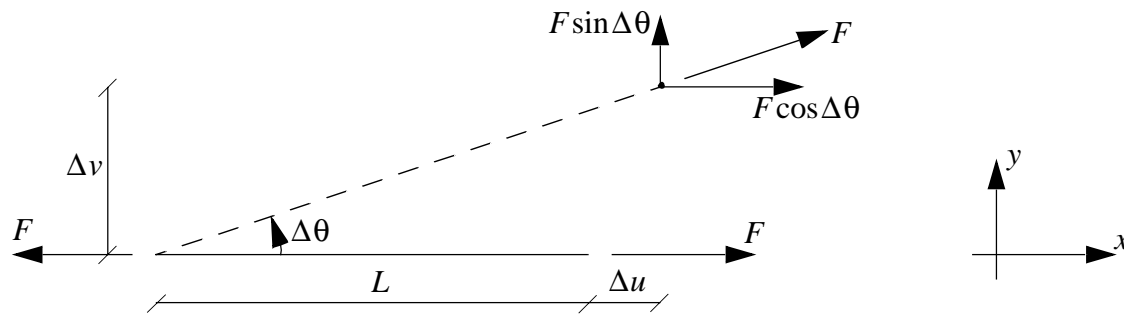$$\underline{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{Identity matrix}$$

So

$\left(\frac{AE_T}{L}\right)\underline{\beta}^T\underline{\beta}$ is the pseudo linear stiffness based on the deformed position defined by $\underline{\beta}$ or $\underline{\alpha}$.

and

$\frac{F}{L}\underline{I}$ is the "geometric" stiffness due to initial forces.

4.2.4 Illustration - Geometric stiffness associated with an initial force

$$\Delta \underline{u} \;=\; \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

Geometric term of $\; d\underline{P} \;=\; \dfrac{F}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$

$$d\underline{P} \;=\; \frac{F}{L}\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

This can also be done by inspection

$$d\underline{P} \;=\; \begin{bmatrix} F\cos\Delta\theta - F \\ F\sin\Delta\theta \end{bmatrix}$$

Assuming small deformations ($e \ll L$)

$$\cos\Delta\theta \;=\; \frac{L+\Delta u}{L+e} \approx 1 + \frac{\Delta u}{L}$$

$$\sin\Delta\theta \;=\; \frac{\Delta v}{L+e} \approx \frac{\Delta v}{L}$$

Therefore

$$d\underline{P} \;=\; \begin{bmatrix} F + F\dfrac{\Delta u}{L} - F \\ F\dfrac{\Delta v}{L} \end{bmatrix} \;=\; \frac{F}{L}\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

4.3 System of Incremental Equilibrium Equations

$$\underline{P}_E = \text{External load vector}$$

$$\underline{P}_{int} = \text{Nodal load vector due to end actions}$$

For Equilibrium

$$\underline{P}_E + (-\underline{P}_{int}) = 0$$

$$\underline{P}_E = \underline{P}_{int}$$

Let $\underline{u}^*$ be an estimate of displacement due to $\underline{P}_E$

$$\underline{P}_{int}(\underline{u}^*) = \text{Load vector (end actions) due to } \underline{u}^* = \underline{P}^*_{int}$$

$$\text{Equilibirum imbalance} = \underline{P}_E - \underline{P}^*_{int}$$

Let $\Delta\underline{u}$ be an estimate of the "correction" required to satisfy equilibrium.

The correction is found with: $\underline{k}^*\Delta\underline{u} = \underline{P}_E - \underline{P}^*_{int}$ where $\underline{k}^*$ is a "predictor" stiffness matrix.

Once $\Delta\underline{u}$ is found, update $\underline{P}^*_{int}$ and $\underline{k}^*$ and use to determine another correction.

This process is repeated until the equilibrium error satisfies the specified convergence criteria.
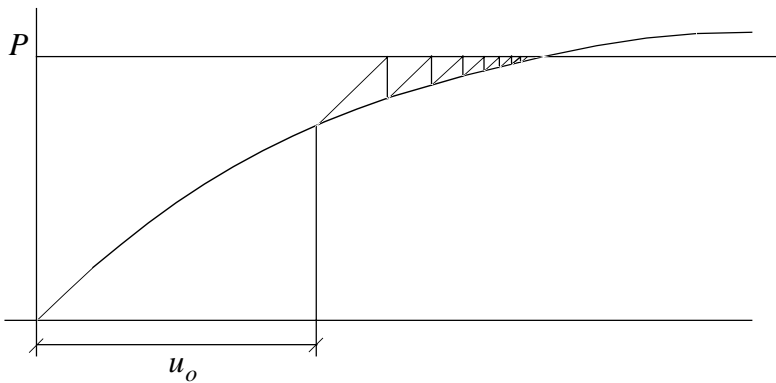
Various choices for $\underline{k}^*$

1) $\underline{k}^* = \underline{k}_{linear}$; the initial stiffness matrix
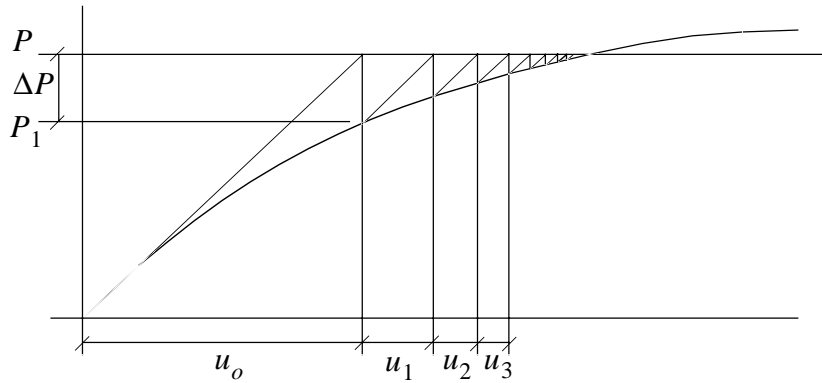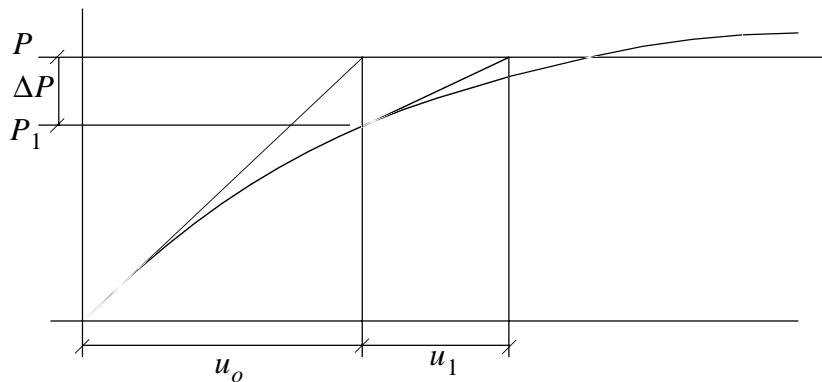

Successive iterations use the initial stiffness

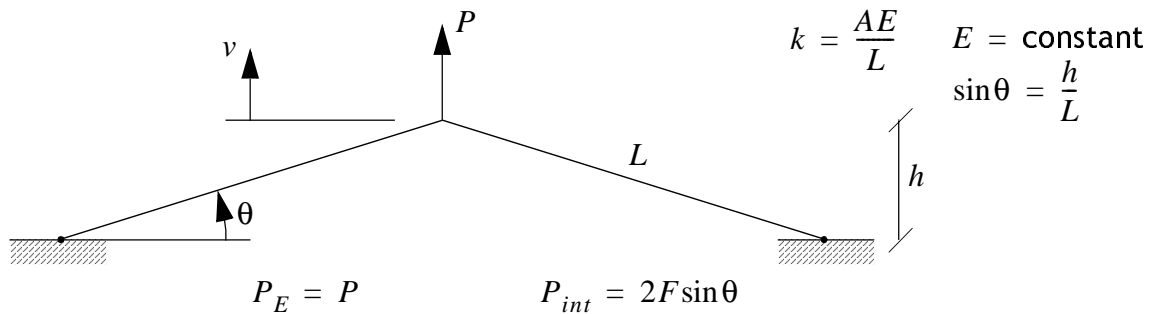Can also start with an inital displacement

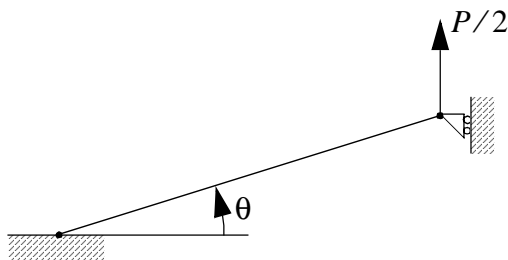2) $\underline{k}^* = \underline{k}_t$; the instantaneous tangent stiffness matrix

Update $\underline{k}_t$ at each iteration. Requires less iteration but is computationally more expensive.

Example using $\underline{k}^* = \underline{k}_t$

$$P_E = P \qquad\qquad P_{int} = 2F\sin\theta$$

Using symmetry

Find the tangent stiffness of the system

$$\underline{\beta} = \underline{\alpha} + \frac{1}{L}\underline{u}^T = \underline{\alpha} + \frac{1}{L}\begin{bmatrix} u & v \end{bmatrix}$$

$$u = 0$$

$$\underline{\alpha} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$\underline{\beta} = \begin{bmatrix} \cos\theta & \dfrac{h}{L} \end{bmatrix} + \frac{1}{L}\begin{bmatrix} 0 & v \end{bmatrix} = \begin{bmatrix} \cos\theta & \dfrac{h}{L} + \dfrac{v}{L} \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \end{bmatrix}$$

$$d\underline{P} = k\underline{\beta}^T\underline{\beta}\Delta\underline{u} + \frac{F}{L}\Delta\underline{u}$$

$$\Delta\underline{u} = \begin{bmatrix} 0 \\ \Delta v \end{bmatrix}$$

$$d\underline{P} = k\begin{bmatrix} \cos^2\psi & \sin\psi\cos\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{bmatrix}\begin{bmatrix} 0 \\ \Delta v \end{bmatrix} + \frac{F}{L}\begin{bmatrix} 0 \\ \Delta v \end{bmatrix}$$

Set $$d\underline{P} = \begin{bmatrix} 0 \\ \Delta P/2 \end{bmatrix}$$

So $$k\sin^2\psi\Delta v + \frac{F}{L}\Delta v = \frac{\Delta P}{2}$$

$$2\left(k\left(\frac{h}{L} + \frac{v}{L}\right)^2 + \frac{F}{L}\right)\Delta v = \Delta P$$

At each interval $k_t(v, F) = 2\left(k\left(\dfrac{h}{L} + \dfrac{v}{L}\right)^2 + \dfrac{F}{L}\right)$ must be updated.

For initial step take

$$\begin{array}{l} F = 0 \\ V = 0 \end{array} \quad \rightarrow \quad k_{to} = 2k\frac{h^2}{L^2}$$

Set

$$k_{to}v_o = P \quad \rightarrow \quad v_o = \frac{P}{k_{to}}$$

and

$$F_o = \frac{P}{2\sin\psi}$$

Update k_t to k_{t1} by including v_o and F_o

Get

$$P_1 = k_{t1}v_o$$

$$P_1 + \Delta P_1 = P \quad \rightarrow \quad \Delta P_1 = P - P_1$$

$$k_{t1}v_o + k_{t1}\Delta v = P$$

Then

$$\Delta v_1 = \frac{P - k_{t1}v_1}{k_{t1}}$$

Repeat until $\Delta v < v_{tol}$ or $\Delta P_n < P_{tol}$

4.4 Linearized Stability Analysis

4.4.1 Bifurcation

The system incremental equation expressed in terms of the instantaneous tangent stiffness matrix

$$\Delta \underline{P} = \underline{K}_t \Delta \underline{u}$$

$\Delta \underline{P}$ = load increment from the equilibrium position ($\underline{P}^*, \underline{u}^*$)

$\Delta \underline{u}$ = first order estimate for displacement increment due to ΔP

Then $\Delta \underline{u}$ can be evaluated using

$$\Delta \underline{u} = (\underline{K}_t)^{-1} \Delta \underline{P}$$

and update $\underline{P}^*, \underline{u}^*, \underline{K}_t$ and $\Delta \underline{P}$

Question: What happens if $\underline{K}_t$ becomes singular?

If $\left| \underline{K}_t \right| = 0$, $\underline{K}_t$ is singular and there exists a non-trivial solution of $\underline{K}_t \Delta \underline{u} = 0$. This implies that there is more than one equilibrium position for a given load.

Suppose $\left| \underline{K}_t \right| = 0$ for $\underline{u}^*$ and $\underline{P}^*$. Then, $\underline{u}^* + \Delta \underline{u}$ is also a solution for the loading $\underline{P}^*$. The existence of multiple equilibrium positions for a given loading is called "bifurcation." In order to determine the structural behaviour in the neighborhood of a bifurcation point, we need to include second and higher order terms in the incremental equilibrium equations. Bifurcation is viewed as an intermediate state. The position is neither stable nor unstable. The terminology "neutral equilibrium" is used to interpret "bifurcation."

4.4.2 Linearized Analysis

Consider the case where the equilibrium position (i.e. the displaced position) is very close to the initial unloaded position. In this case, it is reasonable to neglect the difference between the direction cosines for the initial and deformed configuration. The expression for the member tangent stiffness reduces to

$$\underline{K}_t = k_t \underline{\beta}^T \underline{\beta} + \frac{F}{L} \underline{I} \approx k_t \underline{\alpha}^T \underline{\alpha} + \frac{F}{L} \underline{I}$$

When the material is linear elastic and the member is prismatic $k_t = \dfrac{AE}{L}$

This can be stated

$$\underline{K}_t = \underline{K}_l + \frac{F}{L} \underline{I}$$

Using this result, the system tangent stiffness matrix is expressed as

$$\underline{K}_t = \underline{K}_l + \underline{K}_g$$

where

$\underline{K}_l$ = linear system stiffness matrix

$\underline{K}_g$ = "geometric" stiffness matrix containing the terms involving the memb er forces

$\underline{K}_g$ depends on the loading. As the loading is increased, $\left|\underline{K}_t\right|$ varies from its initial linear value $\left|\underline{K}_l\right|$. Depending on the loading and structural makeup, $\left|\underline{K}_t\right|$ may decrease with increasing load and eventually reach 0, at which point a "bifurcation" occurs. To evaluate whether this is possible, one evaluates $\left|\underline{K}_l + \underline{K}_g\right|$ as a function of the loading.

4.4.2 Example 1
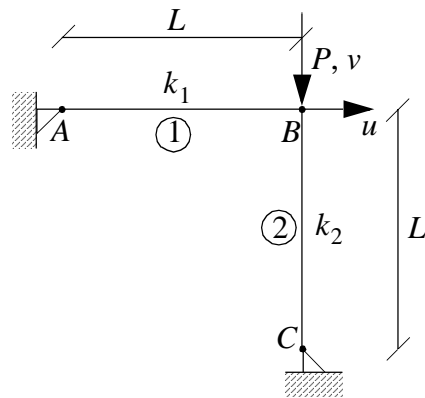
From equilibrium

$$F_2 = -P$$

$$\underline{k}_{t1} \cong \underline{k}_{l1} = \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{k}_{t2} \cong \underline{k}_{l2} + \underline{k}_{g2} = \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \frac{F_2}{L}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{k}_{t2} = \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} -P/L & 0 \\ 0 & -P/L \end{bmatrix}$$

$$\underline{K}_t = \begin{bmatrix} k_1 - P/L & 0 \\ 0 & k_2 - P/L \end{bmatrix}$$

Solve

$$\left|\underline{K}_t\right| = (k_1 - P/L)(k_2 - P/L) = 0$$

Giving

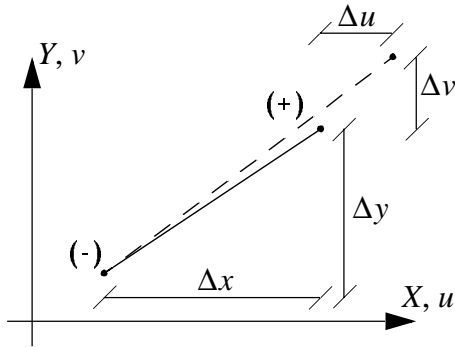For $P = k_1 L$; Δu is arbitrary

For $P = k_2 L$; Δv is arbitrary

4.4.3 Modified Geometric Stiffness Matrix

$$\underline{\beta} = \frac{1}{L+e}(\Delta \underline{x} + \Delta \underline{u})^T \ \text{(exect expression)}$$

For (+) and (-) ends

$\Delta \underline{x}$ = difference in nodal coordinates

$\Delta \underline{u}$ = difference in nodal displacements

$$\Delta \underline{x} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad \Delta \underline{u} = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

Case 1 $\Delta y \cong 0$ (Element ~ parallel to x direction)

Δu will be small wrt Δx

$$\underline{\beta} \cong \frac{1}{L} \begin{bmatrix} \Delta x \\ \Delta y + \Delta v \end{bmatrix}^T$$

Then

$$d\underline{\beta}^T = \frac{\partial \underline{\beta}}{\partial \Delta \underline{u}} \Delta du = \frac{1}{L} \begin{bmatrix} 0 \\ \Delta dv \end{bmatrix}^T$$

and

$$F d\underline{\beta}^T = \frac{F}{L} \begin{bmatrix} 0 \\ \Delta dv \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & F/L \end{bmatrix} \begin{bmatrix} \Delta du \\ \Delta dv \end{bmatrix}$$

Case 2 $\Delta x \cong 0$ (Element ~ parallel to y direction)

Δv will be small wrt Δy

$$\underline{\beta} \cong \frac{1}{L} \begin{bmatrix} \Delta x + \Delta u \\ \Delta y \end{bmatrix}^T$$

Then

$$d\underline{\beta}^T = \frac{1}{L} \begin{bmatrix} \Delta du \\ 0 \end{bmatrix}^T$$

and

$$F d\underline{\beta}^T = \frac{F}{L} \begin{bmatrix} \Delta du \\ 0 \end{bmatrix}^T = \begin{bmatrix} F/L & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta du \\ \Delta dv \end{bmatrix}$$
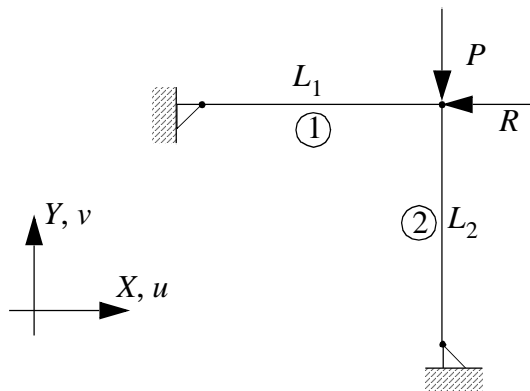
Generalizing

$$k_g = \frac{F}{L}\underline{E}$$

Examine

$$\underline{\alpha} = \begin{bmatrix} \alpha_x & \alpha_y \end{bmatrix}$$

If α_x close to 1, take $\underline{E} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

If α_y close to 1, take $\underline{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

4.4.4 Example 2

$$F_1 = -R$$
$$F_2 = -P$$

$$\underline{k}_{t1} \cong \begin{bmatrix} k_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\dfrac{R}{L_1} \end{bmatrix}$$
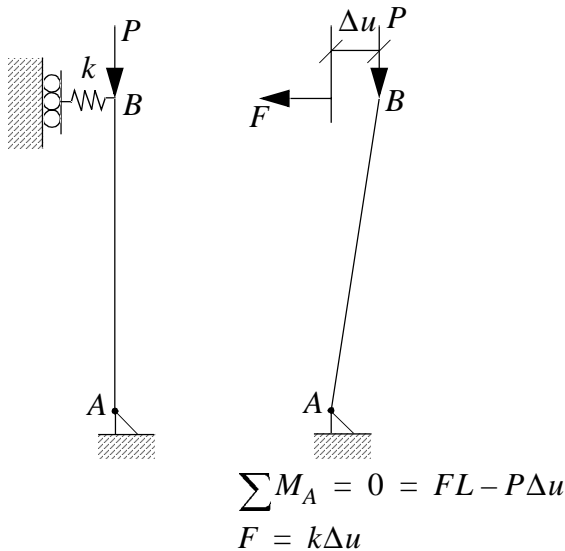
$$\underline{k}_{t2} \cong \begin{bmatrix} 0 & 0 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} -\dfrac{P}{L_2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underline{K}_t = \begin{bmatrix} k_1 - \dfrac{P}{L_2} & 0 \\ 0 & k_2 - \dfrac{R}{L_1} \end{bmatrix}$$

4.4.5 Equilibrium Approach

Perturb structure from its initial equilibrium position, holding the loading constant. Apply equilibrium equations to the perturbed position. Determine whether a non-trivial solution for the "perturbed" displacement is possible. This approach is also seeking whether $|\underline{K}_t| = 0$. However, one does not formally establish $\underline{K}_t$.

Example 3

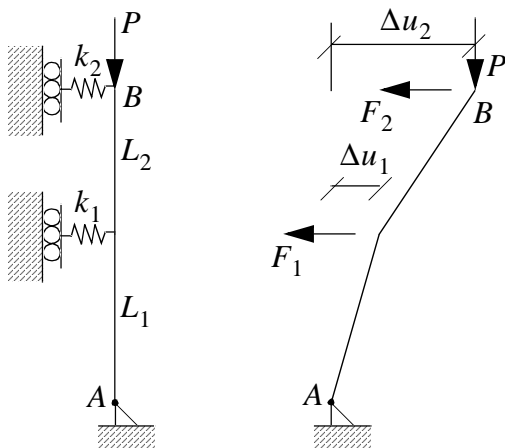$$\sum M_A = 0 = FL - P\Delta u$$
$$F = k\Delta u$$

Then

$$k\Delta u L = P\Delta u$$
$$P_{cr} = kL$$

Example 4

Equilibrium Equations

$$\sum M_{\text{top of } B} = P(\Delta u_2 - \Delta u_1) = F_2 L_2$$
$$\sum M_A = P\Delta u_2 = F_2(L_1 + L_2) + F_1 L_1$$

Member Relations

$$F_1 = k_1 \Delta u_1$$

$$F_2 = k_2 \Delta u_2$$

Then

$$\begin{bmatrix} P & k_2 L_2 - P \\ k_1 L_1 & k_2 (L_1 + L_2) - P \end{bmatrix} \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve by setting detrminant = 0

$$P^2 - P(k_1 L_1 + k_2 (L_1 + L_2)) + k_1 k_2 L_1 L_2 = 0$$

If $k_1 = k_2$ and $L_1 = L_2$

$$\frac{P}{kL} = \frac{1}{2} \{ 3 \pm \sqrt{5} \} = 0.38, 2.62$$

$$\Delta u_2 = \Delta u_1 \left\{ \frac{P}{P - kL} \right\} \quad \leftarrow \quad \text{Mode Shapes}$$

4.4.6 Eigenvalue Approach

For the case where the loading is defined in terms of a single parameter, λ, one can interpret linearized stability analysis as an eigenvalue problem.

Write

$$F = \lambda \bar{F}$$

$$\underline{k}_t = \underline{k}_l + \frac{F}{L} I$$

Set

$$\underline{k}_t = \underline{k}_l - \lambda \underline{k}_g$$

and the equilibrium equations can be written as

$$\underline{k}_l \Delta \underline{u} = \lambda \underline{k}_g \Delta \underline{u}$$