Urban OR: Quiz 2 Solutions (2003)

Problem 1:

(a) $W_o = \sum_{i=1}^{2} \lambda_i \frac{E[S_i^2]}{2} = \frac{12}{60} \cdot \frac{1}{2} + \frac{12}{60} \cdot \frac{9}{2} = 1 \text{ min}$

$$W_{q1} = \frac{W_o}{(1-\rho_1)} = \frac{1}{(1-\frac{12}{60})} = 1.25 \text{ min}$$

$$W_{q2} = \frac{W_o}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{1}{(1-\frac{12}{60})(1-\frac{12}{60}-\frac{12}{20})} = 6.25 \text{ min}$$

$$W_{q,\,priority} = (1.25 + 6.25)/2 = 3.75 \text{ min}$$

$$W_{q,\,FCFS} = W_{q,\,M/G/1} = \frac{\lambda \cdot [\frac{1}{\mu^2} + \sigma_S^2]}{2(1-\rho)} = \frac{\frac{24}{60} \cdot [2^2 + 1]}{2 \cdot (1-\frac{24}{30})} = 5 \text{ min}$$

The expected waiting time for the FCFS system is greater than the expected waiting time for the priority system, as expected. When we assign priority to the customers with the shortest expected service time, we minimize the expected waiting time (see Corollary on page 239 of the text).

(b) Use expression (4.107a) – note that ρ_1=20/60=1/3 and ρ_2=20/20=1 and $\rho_1 + \rho_2$= 4/3 (>1).

$$W_{q,1} = \frac{\rho_1 \cdot \frac{E[S_1^2]}{2 \cdot E[S_1]} + (1-\rho_1)\frac{E[S_2^2]}{2 \cdot E[S_2]}}{(1-\rho_1)} = \frac{\frac{1}{3} \cdot \frac{1}{2 \cdot 1} + \frac{2}{3} \cdot \frac{9}{2 \cdot 3}}{(\frac{2}{3})} = 1.75 \text{ min}$$

and

$$W_{q,2} = \infty$$

(c) Now, since ρ_1=1, neither class of customers attains steady state. Both classes experience infinite expected waiting time in the long run.

Problem 2.

(a) The ambulances are almost always available. The system is either in state 000 or (very rarely) in state 001 or 010 or 100. Thus, whenever an ambulance is dispatched, it is dispatched to a random location in its own district or sector (i.e., the set of points within one mile of the ambulance's home location). Thus the response distance is a r.v. uniform over [0,1]. The mean response distance is 1/2 mile.

(b) The workload or utilization factor is $\rho = \lambda/(3\mu) < 1$. Since $\mu = 1$, $\lambda_{\max} = 3$ calls/hour.

(c) Here all three servers are almost always busy. That is, with probability 1 - *epsilon*, there is always a queue of waiting 'customers.' FCFS means that the location of the next queued customer is uniform over the entire triangle. The triangle circumference is 3 miles. The ambulance takes a shortest route from its home location to the customer. The customer is in one of two 3-mile-length segments emanating from the ambulance's home location, and conditioned on the segment he or she is in, is at a location that is uniformly distributed over that segment. Thus the mean travel distance is 3/2 = 1.5 mi.

(d) The ambulance will travel more than one mile if it is serving an inter-district dispatch. Let P_j = steady state probability the system is in state j, where state j corresponds to j customers in the system. A random customer arrives and finds the system is state j with probability P_j. If the system is in state 0, there is no chance of inter-district dispatch. If the system is in state 1, there is a 1/3 chance of inter-district dispatch. That is, there is a one in three chance that the newly arriving customer is located in the district having the single busy ambulance, thus requiring an inter-district dispatch. If the system is in state 2, following a similar logic, there is a 2/3 chance of an inter-district dispatch. If the system is in state 3 or higher, the customer is queued and by FCFS is handled by an inter-district dispatch with probability 2/3. So, the answer is

$$B_I = 0\, P_0 + (1/3)\, P_1 + (2/3)*(P_2 + P_3 + P_4 + \ldots),$$

B_I = where probability that a random assignment is an inter-district assignment.

By straightforward birth and death queue analysis of Chapter 4, we can find the steady state probabilities:

$$P_0 = [1 + \lambda/\mu + (1/2)(\lambda/\mu)^2/(1 - \lambda/3\mu)]^{-1}$$

$$P_1 = (\lambda/\mu)P_0$$

$$P_2^+ \equiv P_2 + P_3 + P_4 + \ldots = (1/2)(\lambda/\mu)^2/(1 - \lambda/3\mu)P_0$$

Problem 3.

(a) We need a minimal length pair-wise matching of nodes of odd degree, in order to make a new network or graph having all nodes of even degree. Then we can construct an Euler tour. By inspection the matching corresponds to appending to the original network duplicate links for all interior bridges. This is because the bridges are by far the shortest length links that we can use to create an augmented network having all even degree nodes. In practice, this means that the jogger, when approaching a bridge that he has not yet jogged across, would jog across it and then immediately make a U turn and jog back across it. (No need to match the two bridges on the two far ends of the total jogging route, as they have nodes of even degree; all others have nodes of odd degree.) The total length of the jog is then 18.27 + 1.57 = 19.84 miles.

Note 1: One student came up with another way to implement the Euler tour, one that is much less boring from a jogger's point of view. Start at the Science Museum north and jog south along Science Museum land bridge; jog to Longfellow and jog across it; jog to Mass Ave Bridge and jog across it, … continue this alternating path until jogger reaches south end of Watertown Sq. Bridge and jogs across it; now return on the Cambridge side and again jog across each bridge you come across. It works! Draw a picture!

Note 2: Another student noted quite correctly that we did not explicitly ask for an Euler *tour* in part (a), only an Euler *path*. Thus, allowing for Mike Jogger to end up at a different location than his starting location, we do not have to add the longest bridge – the Longfellow Bridge – to our augmented graph. We can allow in the augmented graph two nodes of odd degree. The jogger can start at one end of the Longfellow Bridge and finish his jogging path at the other end. Full credit was given for both interpretations.

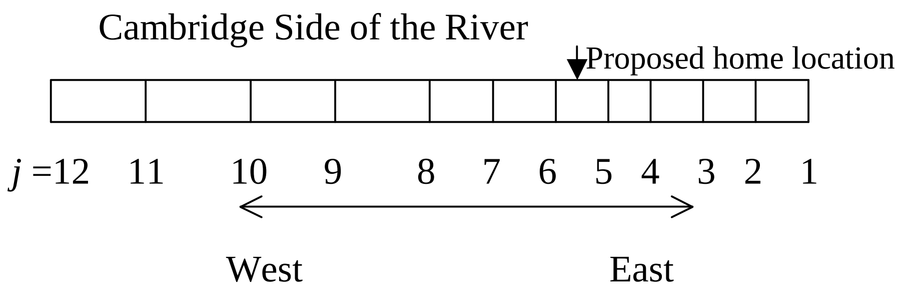
(b) There are $12^2 = 144$ equally likely jogging tours. Note that tour (3,7) for instance is different from tour (7,3); the same path is followed around the two selected bridges, but in reverse directions. On any given day tour *i* occurs with probability 1/144. To construct the probability law for the total jogging distance on a random day, we just compute the jogging length (in miles) for each of the possible 144 tours and assign the probability 1/144 to each. Note that the distance for any tour must include the extra distance, if any, required for the jogger to get to and from the Operations Research Center at MIT to the closer of the two bridges in his tour. We can express the result of combining 144 such calculations either as a probability mass function or as a cumulative distribution function. Suppose the total jogging distance for tour (i,j) is d_{ij}. Then the expected jogging distance on a random day is

$$E[D] = \sum_{i=1}^{12}\sum_{j=1}^{12} d_{ij} / 144$$

(c) Each jogging tour has a minimal jogging distance corresponding to the sum of the two bridge lengths plus the two land distances between the two bridges crossed. If the home location of the jogger on the Cambridge side of the river is between the two bridges crossed, then that location adds zero additional mileage to the route.

However, if that home location is not between the two bridges, then there is a total ‘deadheading’ distance equal to twice the distance from his home location to the closer of the two bridges. This is extra distance he has to jog to get to and return from the cyclic route connecting the two bridges selected for that day. To minimize the expected distance jogged per day, the jogger has to minimize (twice) the expected deadheading distance.

Suppose the jogger’s proposed home location is west of bridge j and east of bridge j+1, j =1,2, …, 11, with the bridges sequentially numbered from 1 to 12, starting with the Science Museum land bridge (at j = 1). In the figure below, j = 5.



That is, the proposed home location is on the Cambridge side between bridges j and j+1. Then there are j^2 deadheading routes east of the home location and $(12-j)^2$ deadheading routes west of the home location. Why? The remaining $2j(12-j)$ tours have no deadheading distance. Why? These observations reduce the problem to a 1-median problem on a straight line. At any proposed home location for the jogger, there are 'nodal weights' totaling j^2 /144 to the east and $(12-j)^2/144$ to the west. For instance if $j^2/144 > (12-j)^2/144$, then the mean daily jogging distance is reduced in a linear manner as one moves the jogger’s home location closer to bridge j and farther from bridge j+1. One seeks a home location for the jogger at which the 'weights' pulling left and right are equal. For if they are not equal, one reduces mean deadheading distance by moving the home location of the jogger in the direction having greater total weight. The optimal balance occurs in this problem at any point between bridge 6 (Western Avenue Bridge) and bridge 7 (Weeks Footbridge), including the two nodes corresponding to the respective end points of those bridges. At these points the two sets of weights are equal, each being 36/144. We recall in the one median that we may have non-nodal optimal locations (in addition to nodal ones) if the weights pulling in each direction are equal. If the total number N of bridges in this problem had been an odd number rather than an even number, then the optimal home location would be at a node equal to the bridge number $(N+1)/2$. For instance, if there had been N = 13 rather then 12 bridges in this problem then the optimal location would be at the node corresponding to bridge (13 + 1)/2 = 7 (Weeks Footbridge). Just as in the regular 1-median location problem, one does not risk missing an optimal solution to the problem by examining possible home locations solely at the nodes. This is true whether there is an even or an odd number of bridges in the problem.

Note: *In doing the quizzes, many students asserted that this is a 1-medan problem, but few argued why that is so and what the objective function is. Few recognized that it is twice the mean deadheading distance that we are attempting to minimize and that the node weights – for tours that do not contain the home location -- are 1/144, not 1/12.*