LECTURES # 35 & # 36

1.060 ENGINEERING MECHANICS II

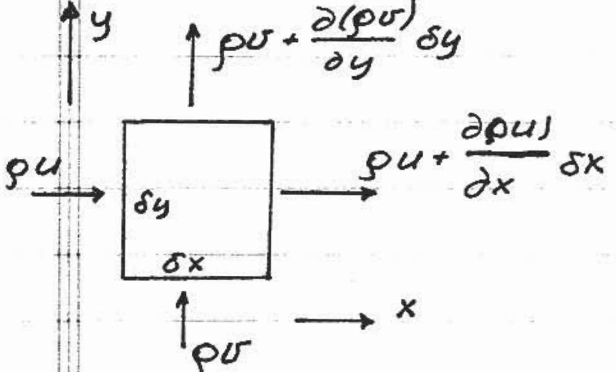
DIFFERENTIAL ANALYSIS OF FLUID FLOWS

THE NAVIER-STOKES EQUATION

Up to this point we have formulated fluid mechanics problems using FINITE CONTROL VOLUME formulations. As a result of this HYDRAULICS approach to problem formulation we have developed an ability to determine "bulk" characteristics of fluid flow variables and effects, e.g. $V = Q/A$, F = total force = $MP_1 - MP_2$, etc. However, we have not resolved the details of the fluid flow, e.g. $v = v(z)$, $p(x, y, z)$, etc. To develop this ability, we must replace the finite control volume approach by an infinitessimal control volume approach, i.e. obtain a description of fluid flow in terms of DIFFERENTIAL EQUATIONS!

Since a fluid is merely a very special type of solid (and far more interesting, since it is "moving") there is a lot of similarity between the derivation of the governing equations for fluids and solids (as covered in 1.050) and we shall take advantage of this.

Conservation of Mass and Volume



We consider a fixed representative elementary volume (REV) $\delta \forall = \delta x \delta y$, and limit ourselves to two-dimensions (unit length into paper).

Conservation of mass states that rate of change of mass within REV = net rate of mass inflow to the REV, or

$$\frac{\partial}{\partial t}(\rho \delta \forall) = \left[\rho u - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\delta x\right)\right]\delta y + \left[\rho v - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\delta y\right)\right]\delta x$$

or
$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} = -\left(\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right)$$

This may be written as

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (1)$$

where

$D\rho/Dt = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}$ = rate of change in ρ of a fluid particles as it moves about.

For fluid = water ρ is a weak function of pressure and temperature, i.e. ρ of a fluid particle is $\approx$ constant, and therefore

$$\frac{D\rho}{Dt} \simeq 0 \qquad (2)$$

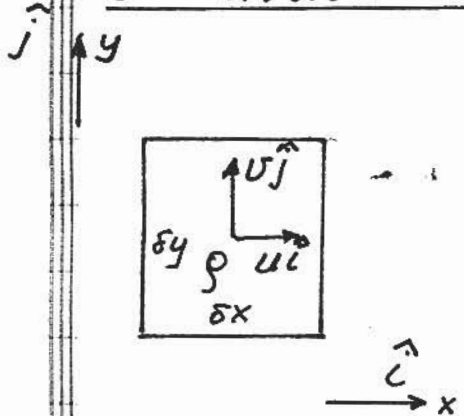
leaving us with

Conservation of volume

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3)$$

valid for an incompressible fluid.

Conservation of Momentum



Newton's Law states that Rate of change of momentum of a fluid particle = sum of forces acting on the fluid particle.

Since Newton's Law applies to a "particle" the rate of change we are looking for is the rate of change following the particle, i.e.

$$\frac{D}{Dt}(\text{Momentum of particle}) = \frac{D}{Dt}\left(\rho\,\delta V(u\hat{i} + v\hat{j})\right) =$$

$$(u\hat{i} + v\hat{j})\frac{D(\rho\,\delta V)}{Dt} + \rho\,\delta V \frac{D}{Dt}(u\hat{i} + v\hat{j}) =$$

$$\rho\left[\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j}\right]\delta V =$$

Sum of forces acting on δV (4)

[The term $\rho\,\delta V$ = mass of fluid particle is taken as constant, i.e. $D(\rho\,\delta V)/Dt = 0$, which actually is an alternative form of conservation of mass]

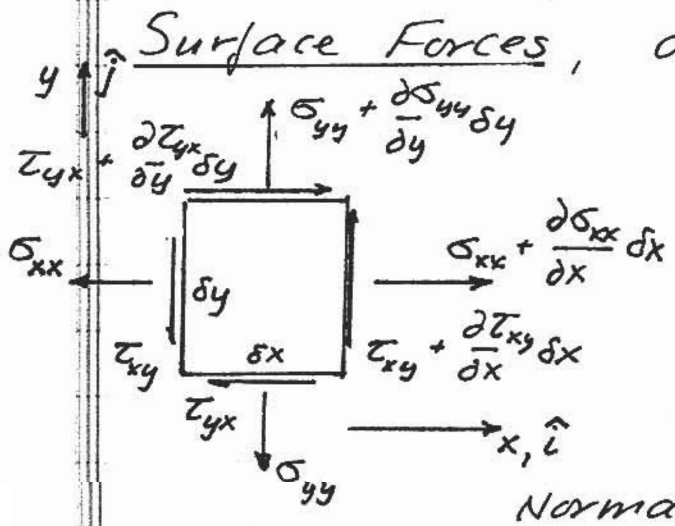
Forces acting on the fluid particle are long range (body forces) and short range (surface forces). Of the former we consider only gravity and have

$$\text{Gravity Force} = \rho\, \delta V \,(g_x \hat{i} + g_y \hat{j}) \qquad (5)$$

where

$$g = \text{Earth's gravitation} \,(= 9.8\, m/s^2) \qquad (6)$$

and g_x and g_y are the components in the x ($\hat{i}$) and y ($\hat{j}$) directions, respectively.



<u>Surface Forces</u>, are expressed in terms of stresses times areas. Following the conventions of solid mechanics we have the ~~fo~~ notation and sign conventions for stresses:

Normal stress: σ; Tangential stress: τ

Subscript and sign convention:

$(\ )_{ij}$: i denotes direction normal to plane upon which stress acts.
j denotes direction of stress

Sign convention: If outward normal is in $\pm$ i-direction stress is positive if acting in $\pm$ j-direction

Note that all stresses indicated on the REV inserted would be considered positive if acting in the direction shown. Also note that by the sign convention of solid mechanics a ~~surfac~~ normal stress is considered positive when it produces <u>tension</u>.

With the aid of the REV and the stresses acting on its surface we obtain:

$$\text{Surface Forces} = \left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)\hat{i} + \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x}\right)\hat{j}\right]\delta x \delta y$$

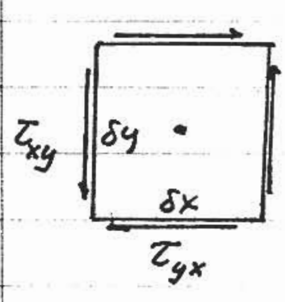
$$= \left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)\hat{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\hat{j}\right]\delta V \quad (7)$$

Introducing (5) and (7) in (4) we obtain the momentum equations for a fluid

$$\rho\left[\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j}\right] =$$

$$\rho\left[g_x\hat{i} + g_y\hat{j}\right] + \left[\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)\hat{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\hat{j}\right] \quad (8)$$

<u>Conservation of Moment of Momentum</u>



Taking the moment of tangential forces acting on the REV around the center of the REV, we obtain

$$\tau_{xy}\,\delta y\,\delta x - \tau_{yx}\,\delta x\,\delta y = \rho I_r\,\ddot{\theta}$$

where I_r is the moment of inertia of the REV and $\ddot{\theta}$ is the angular acceleration.

Since $I_r = \int_{REV} r^2 d\forall \propto \delta x \delta y \cdot O(\delta x^2, \delta y^2)$ it is seen that

$$\underline{\tau_{xy} = \tau_{yx}} \qquad (9)$$

in order to avoid $\ddot{\Theta} \to \infty$ as $\delta x, \delta y \to 0$.

Introduction of the Fluid Pressure

We know that a fluid at rest can support no tangential stresses. In fact, this is the definition of a substance we call a fluid. As a consequence of this, the normal stress condition in a fluid at rest is isotropic, i.e. $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ and ~~σ~~ is independent of direction (orientation) of the plane upon which the normal stress acts.

To preserve this characteristic of a fluid we introduce the concept of fluid pressure defined by

$$(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 = -p \qquad (10)$$

where the "minus sign" signifies that

<u>Fluid Pressure is Positive for Compression</u>

With this definition we may write

$$\sigma_{xx} = -p + \tau_{xx} \, ; \; \sigma_{yy} = -p + \tau_{yy} \, ; \; \sigma_{zz} = -p + \tau_{zz} \qquad (11)$$

where

$$\left.\begin{aligned}\tau_{xx} &= \frac{1}{3}\left(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}\right)\\ \tau_{yy} &= \frac{1}{3}\left(-\sigma_{xx} + 2\sigma_{yy} - \sigma_{zz}\right)\\ \tau_{zz} &= \frac{1}{3}\left(-\sigma_{xx} - \sigma_{yy} + 2\sigma_{zz}\right)\end{aligned}\right\} \quad (12)$$

are normal stresses that vanish when the fluid is at rest, i.e. these normal stresses must be related to the motion (the velocity) of the fluid. Note: τ_{xx} etc are positive for tension!

Clearly, it follows from (10), (11) and (12) that

$$\tau_{xx} + \tau_{yy} + \tau_{zz} = 0 \qquad (13)$$

<u>The Momentum Equation for a Fluid</u>

From the preceding derivations we have the momentum equation in the x-direction by introducing (11) in (8)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right)$$

for the 2-Dimensional case, or for 3-D for

$$\hat{i}: \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \qquad (14a)$$

$$\hat{j}: \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \qquad (14b)$$

and in the z ($\hat{k}$) direction

$$\hat{k}: \quad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \qquad (14c)$$

In Tensor Notation the 3 momentum equations are given by

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \frac{\partial \tau_{ji}}{\partial x_j} \qquad (15)$$

<u>THE NAVIER-STOKES EQUATION</u>

For an incompressible fluid of constant density we have the governing equations (3) and (15), i.e. a total of 4 equations. We should be able to use these equations to solve for 4 variables, e.g.

u, v, w and p

but unless we set all τ_{ij}'s $=0$ [Euler Equations] we are way short of being able to do this.

We have a total of 9 τ_{ij}'s. However, we get some much needed help from the moment of momentum considerations, since (9) generalized to 3-D tells us that

$$\tau_{ij} = \tau_{ji}$$

which produces 3 additional equations.

Finally, we have from the definition of the pressure one more equation:

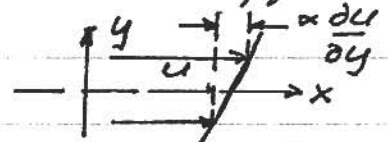
$$\tau_{ii} = \tau_{xx} + \tau_{yy} + \tau_{zz} = 0$$

End result is that we are short a total of $9-3-1 = 5$ equations. Same situation as found in solid mechanics! We need a Stress - Strain relationship - or since we are in a fluid, we need a

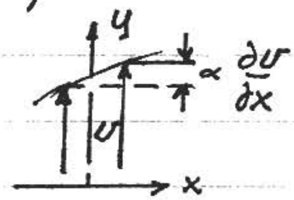
Stress - Rate of Strain Relationship.

In a fluid in motion the shear stress is related to the relative sliding of adjacent layers of fluid. Thus suggest that

$$\tau_{yx} \propto \partial u / \partial y$$



$$\tau_{xy} \propto \partial v / \partial x$$



Since we have $\tau_{yx} = \tau_{xy}$ from (9) this suggest a logical choice to be

$$\tau_{yx} = \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and analogous for the other shear stresses

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right); \quad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

or, in Tensor Notation:

$$\tau_{ij} = \tau_{ji} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{16}$$

The constant of proportionality

$$\mu = \text{dynamic viscosity} \tag{17}$$

is a material (fluid) property

Generalizing (16) also to normal stresses, we obtain

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \; ; \; \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \; ; \; \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

and therefore

$$\tau_{ii} = \tau_{xx} + \tau_{yy} + \tau_{zz} = 2\mu \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \tag{18}$$

as it should be according to (13). This is so only for an incompressible fluid, but since our fluid of choice, water, is incompressible, there is no need to worry about this subtlety.

Thus, we have the general relationship among stresses and strain rates for an incompressible fluid

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{19}$$

Navier-Stokes Equation

Combining the momentum equation and the stress-rate of strain relationship we obtain the Navier-Stokes Equation.

Limiting ourselves to the x-direction we have from (14a) the momentum equation:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

with

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \; ; \; \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) ; \; \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

from (19)

Thus, for a constant viscosity fluid

$$\frac{\partial \tau_{xx}}{\partial x} = 2\mu \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial \tau_{yx}}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial \tau_{zx}}{\partial z} = \mu \frac{\partial^2 u}{\partial z^2} + \mu \frac{\partial^2 w}{\partial x \partial z}$$

$$\sum = \frac{\partial \tau_{ix}}{\partial x_i} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial x \partial y} + \mu \frac{\partial^2 w}{\partial x \partial z} =$$

$$\mu \nabla^2 u + \mu \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = \mu \nabla^2 u$$

since $\partial u_i / \partial x_i = 0$ by virtue of incompressibility.

The Navier-Stokes equations are therefore

in x:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + \mu \nabla^2 u \qquad (20a)$$

and analogous in y:

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + \mu \nabla^2 v \qquad (20b)$$

and z:

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu \nabla^2 w \qquad (20c)$$

where

$$D/Dt = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y + w\,\partial/\partial z$$

and

$$\nabla^2 = \nabla \cdot \nabla = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

for an incompressible fluid for which

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (20d)$$

The 4 equations contain 4 unknowns (velocity components - u, v, w - and pressure - p) and we have what is needed to embark on an exciting journey into Hydrodynamics - BUT WE ARE OUT OF TIME.

GOOD LUCK ON YOUR FURTHER JOURNEY!

MAY THE MOMENTUM BE WITH YOU