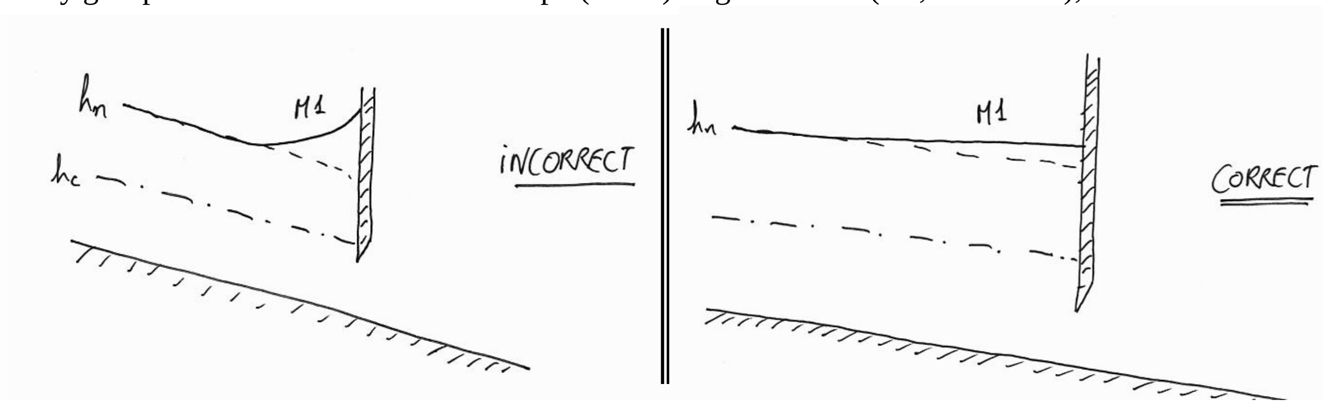
PROBLEM SET 9 - SOLUTIONS

Comments on Problem Set 9

PROBLEMS 1-3:

- A detail that many people got wrong is that the M1 curve tends asymptotically to a horizontal line. Many groups draw the M1 with a final slope (dh/dx) larger than S0 (i.e., horizontal), which is incorrect:



Note that the M1 is a very slowly growing curve, which needs a long distance to yield a significant increase in water depth.

- Our approximate formula to calculate distances (Lecture 30) is an approximation of the equation of the surface profile. Therefore, it only works to calculate distances <u>along a certain gradually varying surface profile</u> (along an M1, along an S3, etc.) You cannot use it along hydraulic jumps. If you have a hydraulic jump (as in problems 2 and 3), you can neglect the distance along it, because this distance is very small (the hydraulic jump is almost vertical). In problem 3, the distance from the gate to vena contracta is also negligibly small. If you have two different curves -e.g., a M2 followed by an S2 in a steep to mild transition- and you need to calculate the total distance, you will have to calculate the distance along M2 and the distance along S2 separately, and add them up.

- Remember that we have an explicit formula to calculate conjugate depths in rectangular channels (see Lecture 28), which is much faster to apply to an unassisted hydraulic jump than to equate the MPs and iterate (the latter procedure is correct, but it takes more time). This formula for conjugate depths will be in the cheat sheet, so you probably want to keep its existence in mind in the exam.

PROBLEM 4:

Most groups did very well on this problem. Remember that there is no C2 curve, and that for normal flow (which coincides with critical flow in this case), dh/dx = 0 (depth of flow doesn't change at all).

PROBLEM 5:

Part b was not counted.

PROBLEM 6:

Everyone did well on this problem. Remember that the specific energy at the critical depth is 3/2*h_c, and that the pressure of a freely falling jet is 0.

PROBLEM SET 9 – SOLUTIONS

– PROBLEM N° 1:

a)

Rectangular channel $\rightarrow \underline{\underline{h_c}} = \left(\dfrac{Q^2}{b^2 g}\right)^{1/3} = \left(\dfrac{10^2}{2^2 \cdot 9'8}\right)^{1/3} = \underline{\underline{1'37\,m}}$

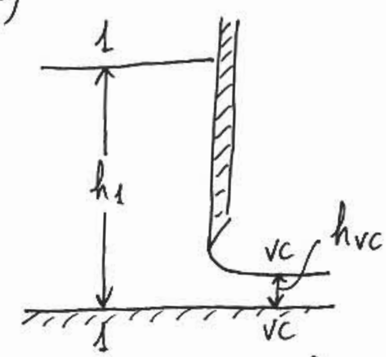
Concrete, finished $\rightarrow n = 0'012$ (Table 10.1)

Get normal depth by iterating in Manning's equation:

$$h_n^{(k+1)} = \left(\frac{Qn}{b\sqrt{S_0}}\right)^{3/5}\left(1 + 2\,\frac{h_n^{(k)}}{b}\right) \text{ ; take } h_n^{(0)} = 0$$

This yields $\underline{\underline{h_n = 2'39\,m}} > h_c \Rightarrow$ MILD SLOPE

b)



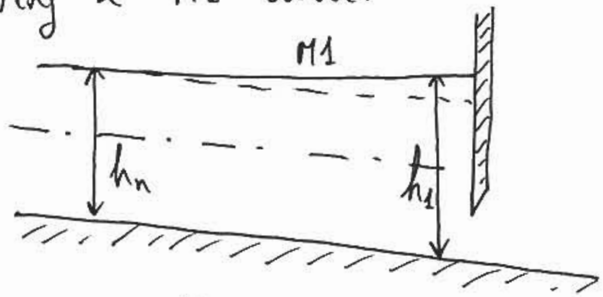
Depth at V.C.: $h_{vc} = C_v \cdot h_g = 0'61$ m.
From 1 to VC we have a short transition of a converging flow, thus energy is conserved.

$$h_1 + \frac{\left(\frac{Q}{b h_1}\right)^2}{2g} = h_{vc} + \frac{\left(\frac{Q}{b h_{vc}}\right)^2}{2g} \Rightarrow h_1 + \frac{1'276}{h_1^2} = 4'038$$

Since $h_{vc} < h_c$ (supercritical), the alternate depth h_1 must be subcritical. Therefore, we have to iterate with the elevation term in the LHS:

$$\left.\begin{aligned} &h_1^{(k+1)} = 4'038 - \frac{1'276}{h_1^{(k)2}} \\ &\text{Take } h_1^{(0)} > h_c \text{, i.e., } h_1^{(0)} = 1'5\,m \end{aligned}\right\} \Rightarrow \underline{\underline{h_1 = 3'96\,m}}$$

c) Far upstream the gate we have normal flow, which is subcritical. Flow transitions from h_n to $h_1 > h_n$ following a M1 curve:



$h_n = 2'39\,m$
$h_1 = 3'96\,m$

d)

$$\frac{\Delta h}{\Delta x} \approx \frac{S_0 - \overline{S_f}}{1 - \overline{Fr^2}}$$

$$h = h_n \Rightarrow \begin{cases} S_{f_n} = S_0 = 10^{-3} \\ Fr_n^2 = \dfrac{V_n^2}{g h_n} = 0'187 \end{cases} \quad ; \quad h = h_1 \Leftrightarrow \begin{cases} S_{f_1} = \left(\dfrac{V_1\, n}{Rh^{2/3}}\right)^2 = 3'1 \cdot 10^{-4} \\ Fr_1^2 = \dfrac{V_1^2}{g h_1} = 0'041 \end{cases}$$

$$\overline{S_f} = \frac{10^{-3} + 3'1 \cdot 10^{-4}}{2} = 6'6 \cdot 10^{-4} \quad ; \quad \overline{Fr^2} = \frac{0'187 + 0'041}{2} = 0'114$$

$$\underline{\underline{\Delta x}} \approx \frac{\Delta h\,(1 - \overline{Fr^2})}{S_0 - \overline{S_f}} = \frac{(3'96 - 2'39) \cdot (1 - 0'114)}{10^{-3} - 6'6 \cdot 10^{-4}} \simeq \underline{\underline{4000\,m}}$$

- PROBLEM Nº 2:

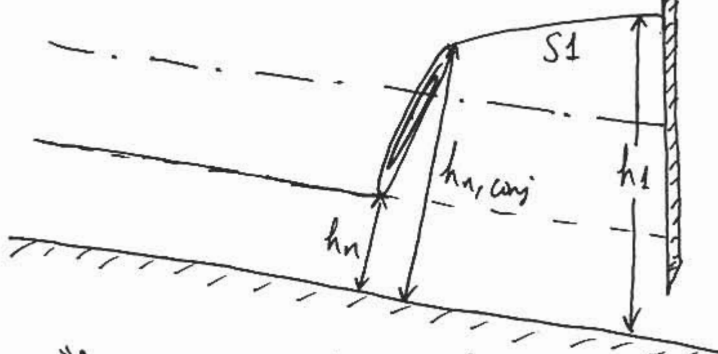
a) $h_c = 1'37\,m$ as before (same Q and same geometry).

Using the same procedure as in problem 1, we get $h_n = 1'25\,m < h_c \Rightarrow$ STEEP SLOPE.

b) The depth upstream the gate is the same as in problem 1, $h_1 = 3'96\,m$.

c) Now, far upstream the gate normal flow is supercritical. To transition from $h_n < h_c$ to $h_1 > h_c$, we need a hydraulic jump from h_n to $h_{n,\,conj}$.

$$h_{n,\,conj} = \frac{h_n}{2}\left[-1+\sqrt{1+8Fr_n^2}\right] = \frac{1'25}{2}\left[-1+\sqrt{1+8\,\frac{(5/1'25)^2}{9'8\cdot 1'25}}\right] = 1'49\,m$$



$h_n = 1'25\,m$
$h_{n,\,conj} = 1'49\,m$
$h_1 = 3'96\,m$

d) The "horizontal" distance along the hydraulic jump between h_n and $h_{n,\,conj}$ is negligible. Therefore, the upstream distance affected by the gate is the distance along the curve S1 between $h_{n,\,conj}$ and h_n.

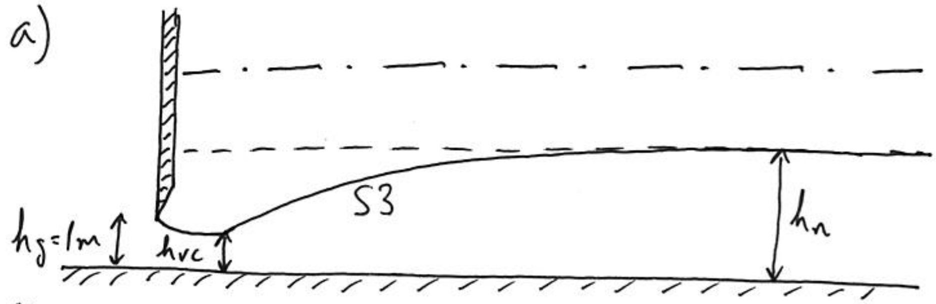
$$h = h_{n,\,conj} \Rightarrow \begin{cases} S_f = 3'2\cdot 10^{-3} \\ Fr^2 = 0'771 \end{cases}$$

$\bar{S_f} = 1'8\cdot 10^{-3}$; $\overline{Fr^2} = 0'406 \Rightarrow \underline{\Delta x \simeq 450\,m}$

– PROBLEM Nº 3:

$\boxed{\text{I}}$

a)



$h_{vc} = 0'61 m$
$h_n = 1'25 m$

b) Distance from gate to V.C. is negligible.

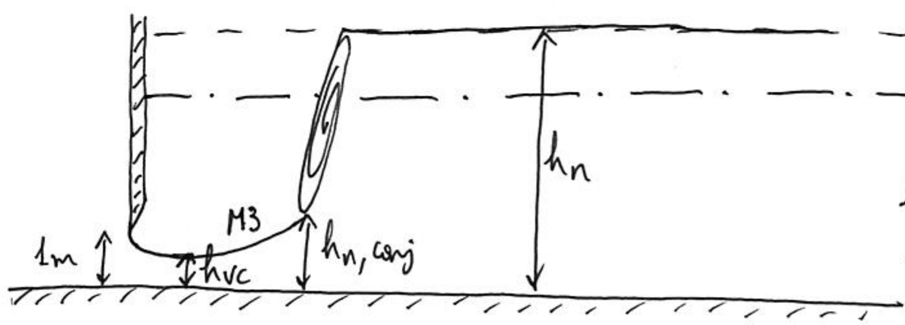
$h = h_{vc} \Rightarrow \begin{cases} S_f = 3'5 \cdot 10^{-2} \\ Fr^2 = 11'24 \end{cases}$ $\quad h = h_n \Rightarrow \begin{cases} S_f = 5 \cdot 10^{-3} \\ Fr^2 = 1'31 \end{cases}$ $\quad \overline{S_f} = 0'02$, $\overline{Fr^2} = 6'28$

$$\underline{\underline{\Delta x}} = \frac{\Delta h(1 - \overline{F_r^2})}{S_o - \overline{S_f}} = \frac{(1'25 - 0'61)(1 - 6'28)}{5 \cdot 10^{-3} - 0'02} \approx \underline{\underline{200 \, m}}$$

$\boxed{\text{II}}$

c) Now we have a transition from $h_{vc} = 0'61 m < h_c$ to $h_n = 2'39 m > h_c$ through a hydraulic jump. The hydraulic jump takes place from $h_{n,conj}$ to h_n.

$$h_{n,conj} = \frac{h_n}{2}\left[-1 + \sqrt{1 + 8Fr_n^2}\right] = 0'69 m > h_{vc} \checkmark$$



$h_{vc} = 0'61 m$
$h_{n,conj} = 0'69 m$
$h_n =$ 2.39 m

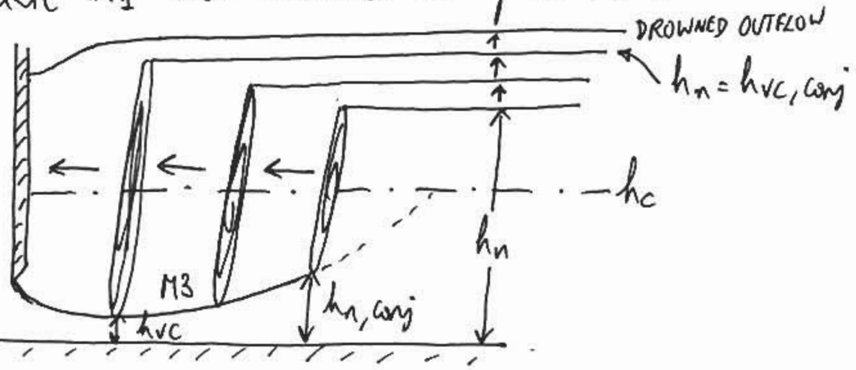
d) We can neglect the distance from gate to VC and from $h_{n,conj}$ to h_n. Thus

$h = h_{vc} \Rightarrow \begin{cases} S_f = 3'5 \cdot 10^{-2} \\ Fr^2 = 11'24 \end{cases}$ $\quad h = h_{n,conj} \Rightarrow \begin{cases} S_f = 2'5 \cdot 10^{-2} \\ Fr^2 = 7'77 \end{cases}$

$\overline{S_f} = 3 \cdot 10^{-2}$; $\overline{Fr^2} = 9'51 \Rightarrow \underline{\underline{\Delta x \approx 30 \, m}}$

e)

If we decrease the slope, h_n will increase, $h_{n,conj}$ will decrease, and the hydraulic jump will move upstream. For a slope such that $h_n = h_{vc,conj}$, the hydraulic jump will exactly happen at the vena contracta location. For a slope even smaller than this one, we will have a drowned outflow. (Note: If drowned outflow, we would have to recalculate h_1 and answers to problem 1 would change!)



- PROBLEM N° 4

Critical slope $\Leftrightarrow h_n = h_c$ — wide rectangular channel

Chezy: $Q = A\,C\sqrt{R_h \cdot S_f} \Rightarrow q = \frac{Q}{b} = h\,C\sqrt{S_f} \Rightarrow S_f = \frac{q^2}{C^2 \cdot h^3}$ (1)

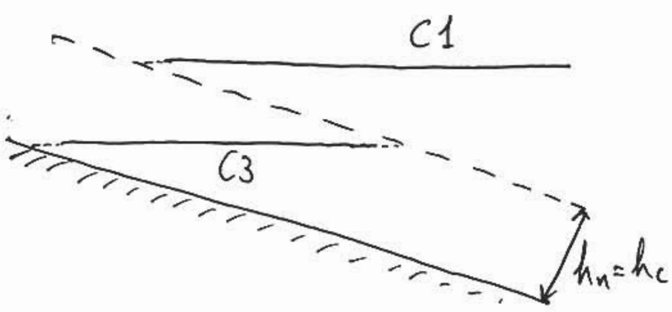
If $h = h_n$, $S_f = S_0 \Rightarrow S_0 = \frac{q^2}{C^2 h_n^3}$ (2) ; $\frac{(1)}{(2)} \Rightarrow \frac{S_f}{S_0} = \left(\frac{h_n}{h}\right)^3$

Froude number: $F_r^2 = \frac{(q/h)^2}{gh} = \frac{q^2}{gh^3}$ (3)

$F_{rc}^2 = 1 = \frac{q^2}{gh_c^3}$ (4)

$\frac{(3)}{(4)} \Rightarrow F_r^2 = \left(\frac{h_c}{h}\right)^3$

$$\frac{dh}{dx} = \frac{S_0 - S_f}{1-F_r^2} = \frac{S_0(1 - S_f/S_0)}{1-F_r^2} = \frac{S_0\left(1-(h_n/h)^3\right)}{1-(h_c/h)^3} \overset{h_n = h_c}{=} S_0$$



Since $h_n = h_c$, no C2 profile. C1 & C3 are horizontal.

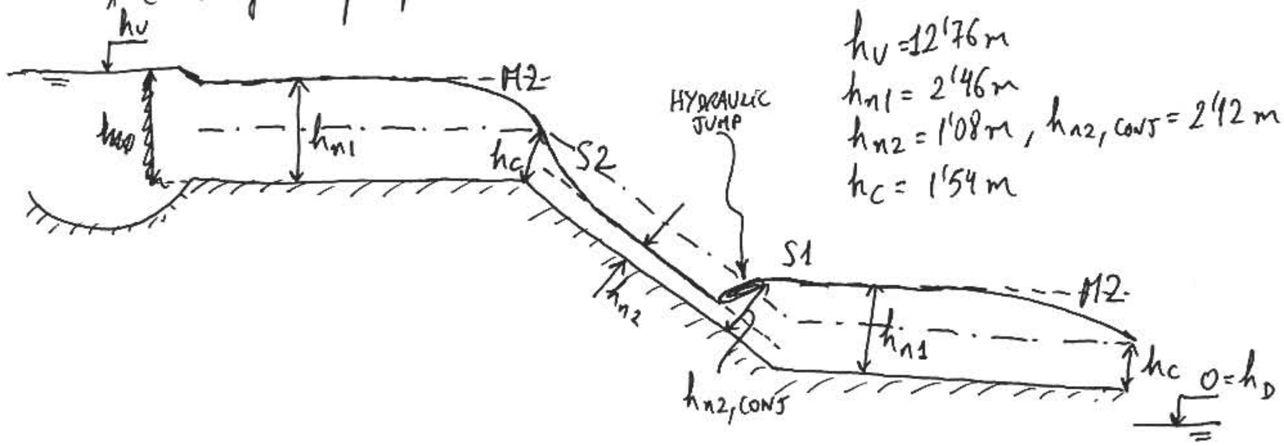
-PROBLEM N° 5 :

a) Calculate h_n, h_c, $h_{n,conj}$ for each stretch:

STRETCH	S_0	h_n	h_c	$h_{n,conj}$
1 (& 3)	0'001	2'46 m	1'54 m	0'89 m
2	0'01	1'08 m	1'54 m	2'12 m

Applying conservation of energy between the upstream lake and the first stretch, we have $\underline{h_U} = 10 + h_{n1} + \frac{V_{n1}^2}{2g} = \underline{\underline{12'76\,m}}$.

Since $h_{n2,conj} < h_{n3} = h_{n1}$, the transition from super- to subcritical flow will happen through an S1 curve near the end of the second stretch. The sketch of the surface profile is:



b)

The length of the last stretch is $L = \frac{\Delta z}{S_{0,1}} = \frac{2-1}{0'001} = 1000\,m$.

The conditions imposed by the lake will affect the stretch of slope $S_{0,2}$ if the distance necessary to transition from h_E (at the end of stretch 3) to $h_{n,3} = h_{n,1}$ is larger than L. In this case, $h < h_{n,1}$ at the beginning of stretch 3 and this will influence the free surface in the intermediate stretch.

- Does this already happen for the initial condition, $h_D = 0$?

$$h = h_{n,1} = 2'46\,m \Rightarrow \begin{cases} S_f = 10^{-3} \\ Fr^2 = 0'247 \end{cases} \quad h_E = h_c = 1'54\,m \Rightarrow \begin{cases} S_f = 3'64 \cdot 10^{-3} \\ Fr^2 = 1 \end{cases}$$

$\overline{S_f} = 2'32$; $\overline{Fr^2} = 0'624 \Rightarrow \Delta x \simeq 250\,m < 1000\,m$. It doesn't.

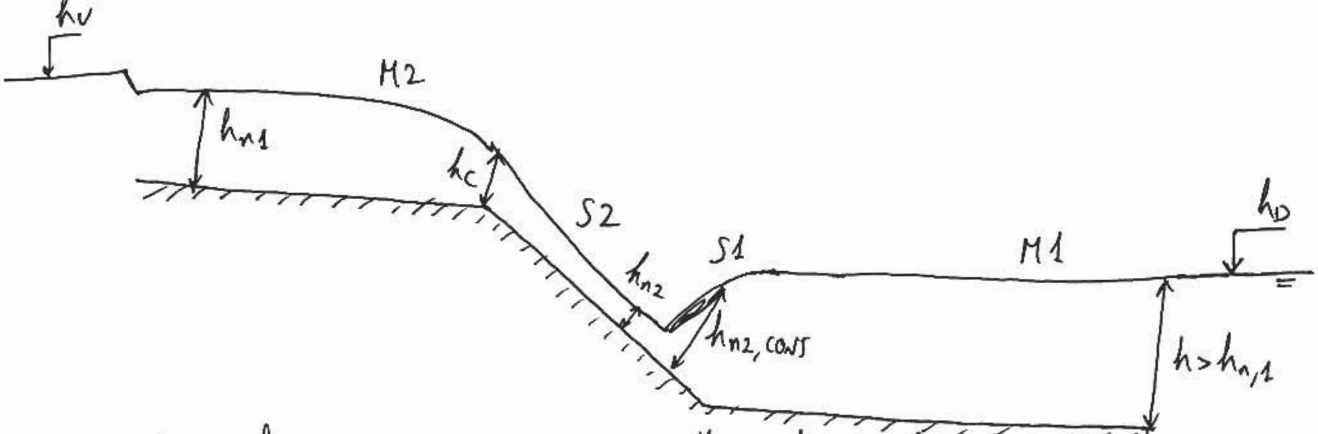
- For $h_D < 1 + h_c = 2'54\,m$, the M2 curve in the last stretch ends at $h = h_c$ (Solution for part a holds)
- For $1 + h_c = 1'54\,m < h_D < 1 + h_{n,1} = 3'46\,m$, we still have a M2 curve, but it ends at h_E ($h_c < h_E < h_{n,1}$). Since h_E is more similar to $h_{n,1}$ than h_c, $\Delta x < 250\,m$, and the M2 curve doesn't influence the intermediate stretch.
- For $h_D > 1 + h_{n,1} = 3'46\,m$ (i.e., $h_E > h_{n,1}$), we get a M1 curve in the last stretch. The M1 is a very slow curve, and it immediately influence the intermediate stretch.

(You can check that taking $h_E = 2'50\,m$, only slightly larger than $h_{n,1}$, which yields $\Delta x \approx 1600\,m > 1000\,m$).

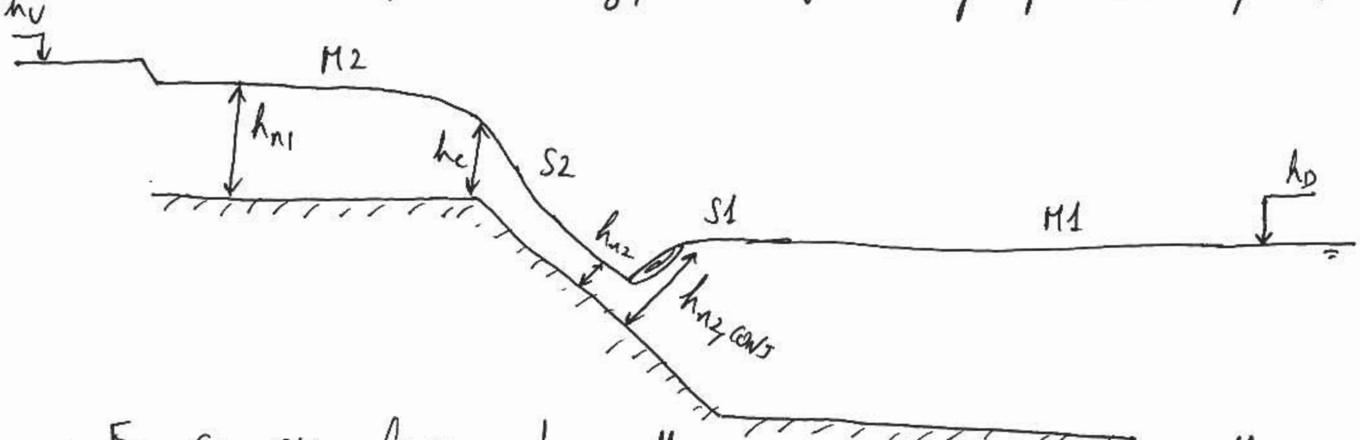
Therefore, $\underline{\underline{h_{D1} \approx 3'5\,m}}$.

c)

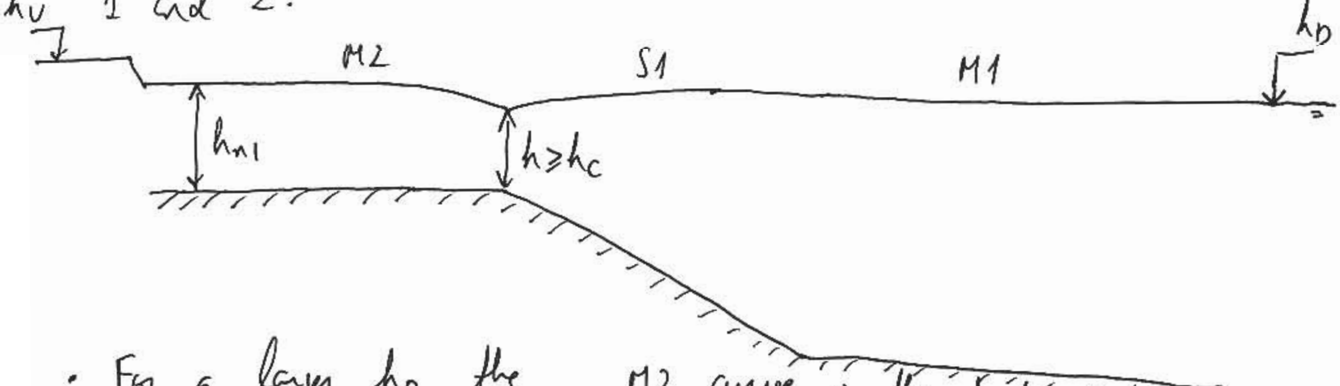
- For $h_D > 1 + h_{n,1}$, the depth at the end of the last stretch will be $h > h_{n,1}$, and we will have an M1 curve:



- When h_D keeps increasing, the hydraulic jump moves upstream:

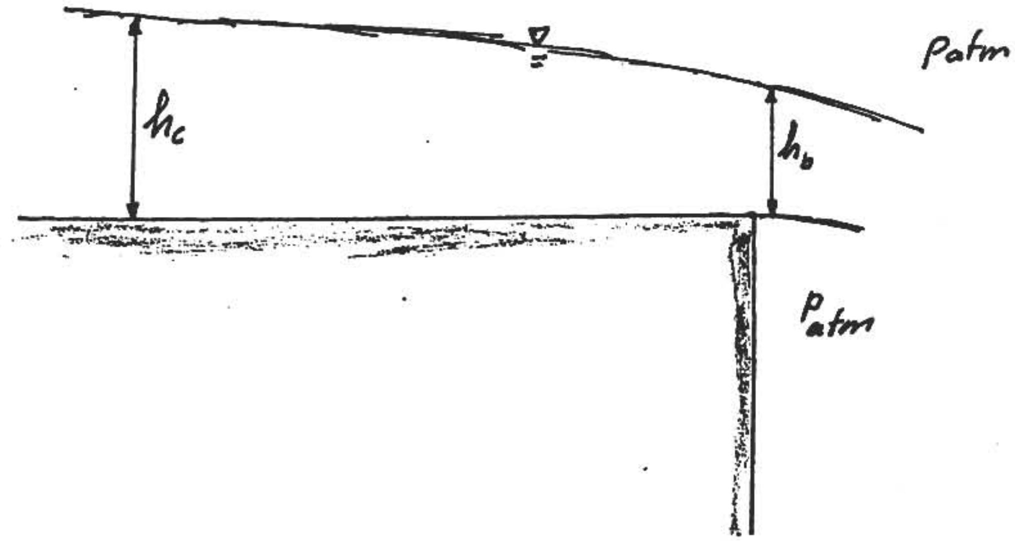


- For an even larger h_D, the curve S2 disappears and the curves M2 and S1 meet in the transition between stretches 1 and 2:



- For a larger h_D, the M2 curve in the first stretch turns into an M1, and we have M1–S1–M1. Eventually, h_V will have to increase in order to keep the discharge Q constant.

Problem No: 6



a)

Since flow approaches the brink in a mildly sloping channel, and the flow after the brink [bottom is "vertical"] is the ultimate in terms of a steep slope, we have a transition from mild to steep slope ⇒ <u>Flow passes through critical "at" transition, i.e. near the brink.</u>

b)

Since channel is rectangular we have $Fr = V/\sqrt{gh}$.
For critical flow, therefore

$$Fr = 1 = V_c/\sqrt{gh_c} \Rightarrow \underline{V_c = \sqrt{gh_c}}$$

$$q = Vh = V_c h_c = \sqrt{g}\, h_c^{3/2} \quad \underline{h_c} = \sqrt[3]{q^2/g} = \left(\frac{3.13^2}{9.8}\right)^{1/3} = \underline{1.00\,m}$$

$$\underline{E_c} = h_c + \frac{V_c^2}{2g} = h_c + \frac{1}{2}h_c = \underline{1.5\,m}$$

c)

$$MP_c = MP_b \quad \text{(short distance, friction may be set=0)}$$

$$MP_c = (\rho V_c^2 + p_{ca,c})\, h_c \cdot 1 = (\rho g h_c + \tfrac{1}{2}\rho g h_c) h_c = \tfrac{3}{2}\rho g h_c^2$$

$$MP_b = (\rho V_b^2 + p_{ca,b})\, h_b \cdot 1 = \rho V_b^2 h_b \quad \text{(since free jet: } p_{ca,b} = 0)$$

$$\tfrac{3}{2}\rho g h_c^2 = \rho (V_b h_b) V_b = \rho q V_b = \rho(\sqrt{gh_c}\, h_c) V_b$$

$$\underline{V_{bM}} = \tfrac{3}{2}\sqrt{gh_c} = 1.5\sqrt{9.8 \cdot 1} = \underline{4.70\ m/s}$$

$$\underline{h_{bM}} = q/V_{bm} = \tfrac{2}{3} h_c = \underline{0.67\,m}$$

d) The "center of gravity" streamline starts at "c" with a total head (measure above bottom) of $H_c = E_c = \frac{3}{2} h_c$

At "b" it has $z_{cg,b} = \frac{1}{2} h_{bB}$, pressure ≈ 0, and velocity V_{bB}, so

$$H_b = \frac{1}{2} h_{bB} + \frac{V_{bB}^2}{2g} = \frac{1}{2} h_{bB} + \frac{(V_{bB} h_{bB})^2}{2g\, h_{bB}^2} = \frac{1}{2} h_{bB} + \frac{q^2}{2g h_{bB}^2}$$

Short transition + Converging Flow $\Rightarrow H_c = H_b$

or with $q = V_c h_c = \sqrt{g}\, h_c^{3/2}$

$$H_c = \frac{3}{2} h_c = H_b = \frac{1}{2} h_{bB} + h_c^3 / 2h_{bB}^2 \Rightarrow \left(3 - \frac{h_{bB}}{h_c}\right)^{1/2} = \frac{h_c}{h_{bB}}$$

Start iteration with $h_{bB}/h_c = 0.67$ from (c)

$$h_{bB}/h_c = 0.65 \Rightarrow \underline{h_{bB}} = \underline{0.65\,m}$$

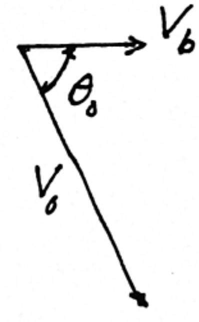
$$\underline{V_{bB}} = q/h_{bB} = \underline{4.80\ m/s}$$

e) Best "estimate" of $V_b = 4.75\ m/s$.

As jet falls it looses no energy since air-resistance may be neglected. Thus, the head at the brink:

$$H_b = H_c = \frac{3}{2} h_c = H_{jet} \text{ everywhere}$$

$$H_{jet} = \frac{V_j^2}{2g} + p_j + z_j = \frac{V_j^2}{2g} + 0 + z_j \quad (p_j \cong p_{atm} = 0)$$



So, $V_j = \sqrt{2g(H_c - z_j)}$

At impact, $z_j \approx -10\,m$, so

$$\underline{V_0} = \sqrt{2g(1.5-(-10))} = \underline{15.0\ m/s}$$

$$V_0 \cdot h_0 = q \Rightarrow h_0 = 3.13/V_0 = \underline{0.21\,m}$$

Without air resistance the initial horizontal velocity at the brink – V_b – is unchanged during free fall. Thus,

$$\cos\theta_0 = V_b/V_0 = 4.75/15 = 0.317 \Rightarrow \underline{\theta_0 = 71.5^\circ}$$