2

# Static Equilibrium Force and Moment

2.1   Concept of Force

Equilibrium of a Particle

You are standing in an elevator, ascending at a constant velocity, what is the resultant force acting on you as a particle?

The correct response is zero: **For a particle at rest, or moving with constant velocity relative to an inertial frame, the resultant force acting on the *isolated particle* must be zero, must vanish.** We usually attribute this to the unquestionable authority of Newton.

The essential phrases in the question are *constant velocity, resultant force* and *particle.* Other words like "standing", "elevator", "ascending", and "you" seem less important, even distracting, but they are there for a reason: The world that you as an engineer will analyze, re-design, and systematize is filled with people and elevators, not isolated particles, velocity vectors, or resultant forces — or at least, not at first sight. The latter concepts are abstractions which you must learn to identify in the world around you in order to work effectively as an engineer, e.g., in order to design an elevator. The problems that appear in engineering text books are a kind of middle ground between abstract theory and everyday reality. We want you to learn to read and see through the superficial appearances, these descriptions which mask certain scientific concepts and principles, in order to grasp and appropriate the underlying forms that provide the basis for engineering analysis and design.

The key phrase in Newton's requirement is ***isolated particle***: It is absolutely essential that you learn to abstract out of the problem statement and all of its relevant and irrelevant words and phrases, a vision of a particle as a point free in space. It's best to render this vision, this abstraction "hard" by drawing it on a clean sheet of paper. Here is how it would look.

An Isolated Particle:

You, in an elevator.

An Isolated Particle
All Forces Acting
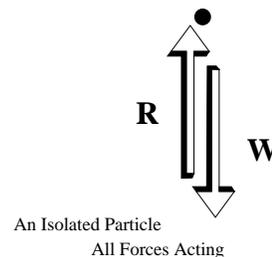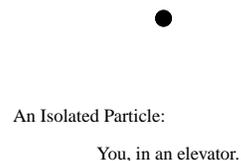
This is a non-trivial step, akin to a one month old's apprehension that there are other egos in the world. You are to take the dot drawn as the representation of a thing, all things, that can be thought of as an isolated particle.

Now show all the forces acting on the particle. We have the force due to gravity, *W=Mg*, acting vertically down, toward the center of the earth.... (Who said the elevator was oriented vertically? Who said it was on the surface of the earth? This information is not given; indeed, you could press the point, arguing that the question is not well posed. But is this information essential? We return to this point at the end of this chapter). We have the *reaction force* of the elevator floor acting vertically upward on you, on you as an abstraction, as an isolated particle. This is how our particle looks with all forces acting upon it.

The *resultant force* is the vector sum of all the forces acting on the isolated particle. For *static equilibrium* of the isolated particle, the resultant of the two forces – *W* acting downward and *R* acting upward – must be zero.

$$R - W = 0$$

This leads to the not very earth shaking conclusion that the magnitude of the reaction force, acting up, must equal the weight.

$$R = W$$

This apparently trivial result and simplicity of the problem, if indeed it can be called a problem, ought not to be allowed to deceive us: The introduction of the reaction of the floor on you, the passenger in the elevator, is characteristic of the most difficult step in applying the requirement of static equilibrium to an isolated particle. You will find it takes courage, as well as facility with the language of engineering mechanics, to venture forth and construct reaction forces out of thin air. They are there, hidden at the interface of your particle with the rest of the world. Some, like gravity, act at a distance, across all boundaries you may draw.

Exercise 2.1

Estimate the lift force acting on the wings of a Boeing 747 traveling from New York to Los Angeles during rush hour.

We can use the same isolation, or ***free-body diagram***, of the figures above where now the point represents the Boeing 747, rather than you in an elevator, and the reaction force represents the lift force acting on the airplane, rather than the force acting on you at your interface with the elevator floor. From the requirement of static equilibrium, (we implicitly acknowledge that the 747 is moving with constant velocity), we conclude that lift force is equal to the weight, so to estimate the lift force we estimate the weight. Constructing this argument is half the problem. Now the other half: To estimate the weight we can guess... 100 tons? (More than 10 tons; we have heard of a 10 ton truck). But perhaps we can do better by slicing up the total weight, and try to estimate its ingredients. Passengers: How many passengers? I estimate 500. (More than 100, less than 1000). Then

$$weight/passenger \approx 250(pounds)$$

where I've thrown in an estimate of the weight of passenger luggage. This gives a contribution of

$$500 \times 250 = 125 \cdot 10^3 (pounds)$$

Structural weight: Let's focus on the weight of the fuselage and the weight of the wings... I imagine the fuselage to be a thin, circular cylinder– a tin can, an aluminum can, a *big* aluminum can. How big? How long, what diameter, what thickness? I will build up upon my estimate of number of passengers and my memory of seating arrangements in the big plane. I estimate 10 or 12 seats across, add another two seats for the two aisles, and taking seat width as two feet, I obtain a cylinder diameter of

$$14 \times 2 = 28 \ (feet)$$

I will round that up to 30 feet. Length: With 14 passengers per row and 500 passengers on board, and taking row pitch as 3 feet/row, I estimate total row length to be

$$3 \times 500 / 14 = 100 (feet)$$

That seems low. But other spaces must be accounted for. For example, galleys and restrooms, (another 40 feet?), and the pilot's compartment, (20 feet), the tail section (20 feet). Altogether then I estimate the length to be 180 feet. Thickness: I estimate 1/2 inch. Now I need a density, a weight density. Water is 62.4 pounds per cubic foot. The specific gravity of aluminum is what? I guess it to be 8. My estimate of fuselage weight is then the volume of the thin cylinder times the weight density. The volume is the circumference times the thickness – a good approximation when the thickness is small relative to the diameter – times the length. I obtain

$$\pi \times 30 \times 180 \times (\ 1/2) \times (1 ft/12 in) \times 8 \ \times 62,4 \ \approx 350 \cdot 10^3 (pounds)$$

Flooring and equipment and cosmetic structure will add more. I add another 20-25% and bring this up to

$$420 \cdot 10^3 (pounds)$$

The wings and engines come next: Here again we estimate the volume of wing material, now taking the wing as the equivalent of two thin sheets of aluminum. Wing length, tip to tip, as approximately the fuselage length, 180 feet. Wing breadth, or mean cord length we take as 20 feet, (the wing tapers as you move out from the fuselage to the tips of the wings) and the thickness again as .5inches. The product of volume and weight density is then, noting that it take two such sheets to make the tip-to-tip wing

$$2 \times 180 \times 20 \ \times (1/2) \times (1/12) \times 8 \times 62, 4 \ \approx 150 \cdot 10^3 (pounds)$$

I throw in another 50,000 pounds for the motors and the tail (plus stabilizers) and so estimate the *total* structural weight to be

$$620 \cdot 10^3 (pounds)$$

Fuel: How much does the fuel weigh? The wings hold the fuel. I estimate the total volume enclosed by the wings to be the wing area times 1 foot. I take the density of fuel to be the same as water, 62.4 pounds per cubic foot. The total weight of the fuel is then estimated to be

$$2 \times 180 \times 20 \times 1 \times 62,4 \approx 450 \cdot 10^3 (pounds)$$

This looks too big. I can rationalize a smaller number citing the taper of the wing in its thickness as I move from the fuselage out to wing tip. I cut this estimate in half, not knowing anything more than that the tip volume must be near zero. So my fuel weight estimate is now

$$250 \cdot 10^3 (pounds)$$

All together then I estimate the lift force on a Boeing 747 at rush hour (fully loaded) to be

$$970 \cdot 10^3 \text{ pounds or approximately 500 tons.}$$

Is this estimate correct? Is it the right answer? Do I get an *A*? That depends upon the criteria used to differentiate right from wrong. Certainly we must allow for more than one numerical answer since there is *no* one numerical answer. If we admit a range, say of 20% either up or down, I may or may not pass. If we accept anything within a factor of 2, I am more confident, even willing to place a bet at 2 to 1 odds, that I am in the right. But go check it out: Jane's **All the World's Aircraft** will serve as a resource.

Is the method correct? The criteria here are more certain: In the first place, it is essential that I identify the lift force as the weight. Without this conceptual leap, without an abstraction of the plane as a particle, I am blocked at the start. This is a nontrivial and potentially argumentative step. More about that later.

Second, my method is more than a guess. It has a rationale, based upon a dissection of the question into pieces – passenger weight, structural weight, fuel weight – each of which in turn I might guess. But, again, I can do better: I dissect the passenger weight into a sum of individual weights. Here now I am on firmer ground, able to construct an estimate more easily and with confidence because an individual's weight is close at hand. So too with the structural and fuel weight; I reduce the question to simpler, more familiar terms and quantities. Fuel is like water in weight. The fuselage is a *big* aluminum can of football field dimensions.

Here I have made a significant mistake in taking the specific gravity of aluminum as 8, which is that for steel. I

ought to have halved that factor, better yet, taken it as 3. My total estimate changes but not by a factor of 2. The method remains correct.

Is this the *only* method, the only route to a rational estimate? No. A freshman thought of the weight of a school bus, fully loaded with forty passengers, and scaled up this piece. A graduate student estimated the lift force directly by considering the change in momentum of the airstream (free stream velocity equal to the cruising speed of the 747) as it went over the wing. There are alternative routes to follow in constructing an estimate; there is no unique single right method as there is no unique, single right number. This does *not* mean that there are *no* wrong methods and estimates or that some methods are not better than others.

Often you will not be able to develop a *feel* for the ingredients of an estimate or the behavior of a system, often because of a disjunction between the scale of things in your experience and the scale of the problem at hand. If that be the case, try to breakdown the system into pieces of a more familiar scale, building an association with things you do have some feel for. More seriously, the dictates of the fundamental principles of static equilibrium might run counter to your expectations. If this is the case, stick with it. In time what at first seems counter-intuitive will become familiar.

Exercise 2.2

What do you need to know to determine the force in cables AB and BC?
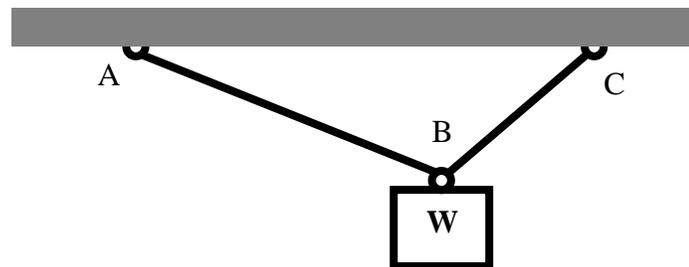
- You *need to know* that the cables support only tension. A *cable* is not able to support *compression* nor does it offer any resistance to *bending*. Webster's *New Collegiate Dictionary* notes the "great tensile strength" of a cable but says nothing about bending or compression. The cable's inability to support other than tension is critical to our understanding, our vocabulary. *Bending* itself will require definition... in time.

- You *need to know* the weight of the block.

- You *need to know* the angles AB and BC each make with the horizontal. (I will call them Θ_A and Θ_B). That's what you *need to know*.

- **You also must know how to isolate the system as a particle and you must know the laws of static equilibrium for an isolated particle.**

- Less apparent, you must presume that the force on the block due to gravity acts vertically downward — the convention in this textbook.

That is the answer to this *need to know* problem.

Now, you might ask me how do I know when to stop; how do I know when I don't need to know anymore? For example, how do I know I *don't* need to know the length of the cables or what material they are made from?

I know from solving this kind of problem before; you will learn when to stop in the same way – by working similar problems. I know too that the materials and lengths of the cables can be essential ingredients in the response to other questions about this simple system...e.g., if I were to ask *How much does the point B drop when the block is hooked up to the cables?* But that question wasn't asked.

Perhaps the best way to decide if you know enough is to try to solve the problem, to construct an answer to the question. Thus:

Exercise 2.3

Show that the forces in cable AB and BC are given by

$$F_C = W\cos\theta_A / \sin(\theta_A + \theta_C) \qquad \text{and} \qquad F_A = W\cos\theta_C / \sin(\theta_A + \theta_C)$$

We first isolate the system, making it a particle. Point *B*, where the *line of action* of the weight vector intersects with the lines of action of the tensions in the cables becomes our particle.

The three force vectors $\mathbf{F_A}$, $\mathbf{F_C}$ and $\mathbf{W}$ then must sum to zero for static equilibrium. Or again, the *resultant force on the isolated particle must vanish*. We meet this condition on the *vector sum* by insisting that two *scalar sums* – the sum of the horizontal (or *x*) components and the sum of the vertical (or *y*) components – vanish independently. For the sum of the *x* components we have, taking positive *x* as positive:

$$-F_A\cos\theta_A + F_C\cos\theta_C = 0$$

and for the sum of the *y* components, $\qquad F_A\sin\theta_A + F_C\sin\theta_C - W = 0$

A bit of conventional syntax is illustrated here in setting the sums to zero rather than doing otherwise, i.e., in the second equation, setting the sum of the two vertical components of the forces in the cables equal to the weight. Ignoring this apparently trivial convention can lead to disastrous results, at least early on in
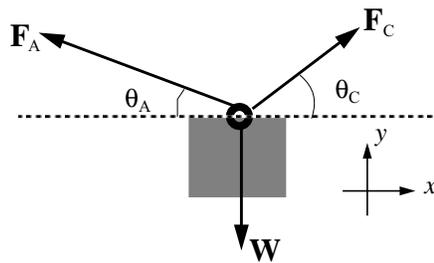
learning one's way in Engineering Mechanics. **The convention brings to the fore the necessity of isolating a particle before applying the equilibrium requirement.**

We see that what we *need to know* to determine the force in cable AB and in cable BC are the angles θ_A and θ_C and the weight of the block, *W*. These are the *givens*; the *magnitudes* of the two forces, F_A and F_C are our two *scalar unknowns*. We read the above then as two scalar equations in two scalar unknowns. We have reduced the problem ... *show that*... to a task in elementary algebra. To proceed requires a certain versatility in this more rarefied language.

There are various ways to proceed at this point. I can multiply the first equation by $sin\theta_A$, the second by $cos\theta_A$ and add the two to obtain

$$F_C \cdot (sin\theta_A \cdot cos\theta_C + sin\theta_C \cdot cos\theta_A) \ = \ W cos\theta_A$$

Making use of an appropriate trigonometric identity, we can write:[1]

$$F_C = \ W \cdot cos\theta_A / sin(\theta_A + \theta_C)$$

Similarly, we find:

$$F_A = \ W \cdot cos\theta_C / sin(\theta_A + \theta_C)$$

And thus we have shown what was asked to be shown. We have an answer, a unique answer in the sense that it is the only acceptable answer to the problem as stated, an answer that would merit full credit. But it is also an answer that has some depth, richness, a *thick* answer in that we can go beyond *show that* to *show and tell* and tease out of our result several interesting features.

- First, note that the derived equations are *dimensionally correct*. Both sides have the same units, that of force (or ML/T^2). In fact, we could easily obtain *non-dimensional* expressions for the cable tensions by dividing by the weight *W*. This *linear* relationship between the unknown forces in the cables and the applied load *W* will characterize most all of our discourse. It is a critical feature of our work in that it simplifies our task: If your boss asks you what will happen if some idiot accidentally doubles the weight hanging from the cables, you simply respond that the tensions in the cables will double. As always, there are caveats: We must assure ourselves first that the cables do not deform to any significant degree under double the weight. If they do then the angle θ_A and θ_C might change so a factor of 2.0 might not be quite exact. Of course if the cables break all bets are off.

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1. This is an example of textbook rhetoric cryptically indicating a skipped step in the analysis. The author's presumption is that you, the reader, can easily recognize what's been left out. The problem is that it takes time, sometimes a long time, to figure out the missing step, certainly more time than it takes to read the sentence. If you are befuddled, an appropriate response then is to take some time out to verify the step.

- Second, note that the more vertically oriented of the two cables, the cable with its θ closer to a right angle, experiences the greater of the two tensions; we say it *carries the greatest load*.

- Third, note that the tension in the cables can be greater than the weight of the suspended body. The denominator $\sin(\theta_A + \theta_C)$ can become very small, approaching zero as the sum of the two angles approaches zero. The numerators, on the other hand, remain finite; $\cos\theta$ approaches 1.0 as θ approaches 0. Indeed, the maximum tension can become a factor of 10 or 100 or 1000... whatever you like... times the weight.

- Fourth, note the symmetry of the system when θ_A is set equal to θ_C. In this case the tensions in the two cables are equal, a result you might have guessed, or should have been able to claim, from looking at the figure with the angles set equal.[2]

- Fifth, if both angles approach a right angle, i.e., $\theta_A \to \pi/2$ *and* $\theta_C \to \pi/2$, we have the opportunity to use "L'Hospital's rule". In this case we have

$$F_C = F_A = \lim_{\theta \to \pi/2} (W\cos\theta / \sin 2\theta) = \lim_{\theta \to \pi/2} (-W\sin\theta / 2\cos 2\theta) = W/2$$

so each cable picks up half the weight.

Other observations are possible: What if one of the angles is negative? What if a bird sits on a telephone wire? Or we might consider the graphical representation of the three vectors in equilibrium, as in the following:

The figure below shows how you can proceed from knowing magnitude and direction of the weight vector and the directions of the lines of action of the forces in the two cables to full knowledge of the cable force vectors, i.e., their magnitudes as well as directions. The figure in the middle shows the directions of the lines of action of the two cable forces but the line of action of the force in cable AB, inclined at an angle θ_A to the horizontal, has been displaced downward.
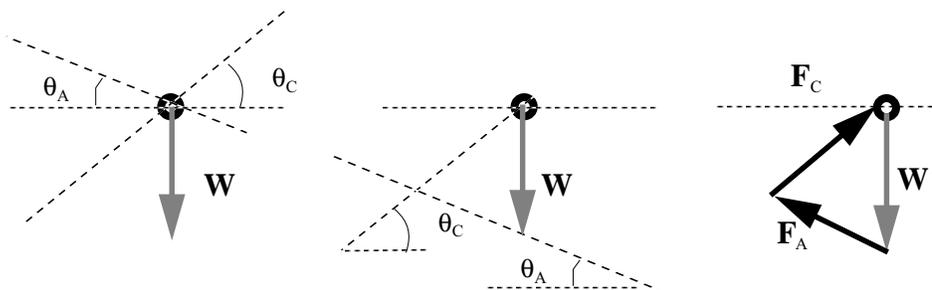
2. Buridan, a medieval scholar in Mechanics would have cited the *Principle of Sufficient Reason* in explaining how the forces must be equal for the symmetric configuration. There is no reason why one or the other cable tensions should be greater or less than the other. Buridan's ass, confronting two symmetrically placed bales of hay in front of his nose, starved to death. There was no sufficient reason to go left or right, so the story goes.

The figure at the far right shows the vectors summing to zero, that is, in vector notation we have:

$$\mathbf{F}_A + \mathbf{F}_C + \mathbf{W} = 0$$

While the two scalar equations previously derived in answering the *show that* challenge do not spring immediately from the figure, the relative magnitudes of the cable tensions are clearly shown in the figure at the far right.

- Graphical and algebraic, or analytic, constructions and readings of problems are complementary. Both should be pursued if possible; beyond two dimensions, however, graphical interpretations are difficult.

- Finally, note how the problem would have differed if the weight *W* and the angles θ_A and θ_C were stated as numbers, e.g., 100 pounds, 30 degrees, 60 degrees. The *setting-up* of the problem would have gone much the same but the solution would be thin soup indeed – two numbers, 50 pounds and 87.6 pounds, and that's about it; no opportunity for real thought, no occasion for learning about medieval scholars thoughts about sufficient reasoning, for conjecturing about birds on telegraph wires or applying L'Hospital's rule; it would be an exercise meriting little more than the crankings of a computer.

Resultant Force

We have used the phrase *resultant* in stating the requirement for static equilibrium of an isolated particle – the *resultant* of all forces acting on the isolated particle must vanish. Often we use *resultant* to mean the vector sum of a subset of forces acting on a particle or body, rather than the vector sum of *all* such forces. For example, we can say "the resultant of the two cable tensions, $\mathbf{F}_A$ and $\mathbf{F}_C$ acting at point *B* in Exercise 2.3 is the force vector **-W**". The resultant is constructed using the so called *parallelogram law for vector addition* as illustrated in the figure. $\mathbf{F}_A + \mathbf{F}_C$ can then be read as **-W**, the vector equal in magnitude to the weight vector **W** but oppositely directed.
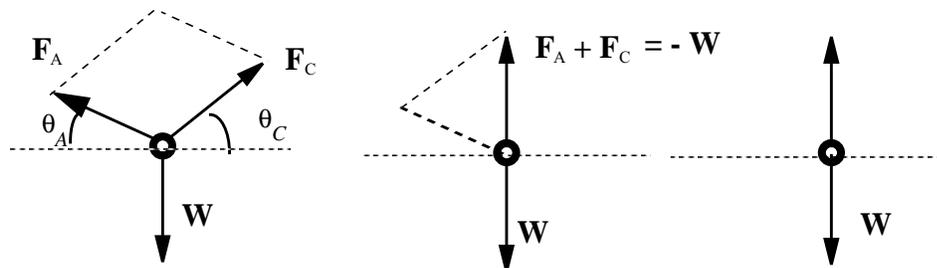
We can also speak of a single force vector as being the *resultant* of its components, usually its three mutually perpendicular (or *orthogonal*) rectangular, cartesian components. In the figure, the vector **F** has components $\mathbf{F}_x$, $\mathbf{F}_y$ and $\mathbf{F}_z$. The latter three mutually perpendicular vectors are usefully written as the product of a scalar magnitude and a unit vector indicating the direction of the vector component. The three unit vectors are often indicated by **i, j** and **k** and that is the convention we will follow in this text.
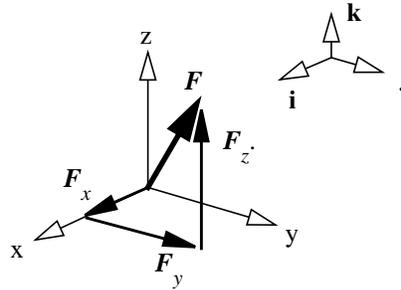
The vector resultant or sum can be written out as

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$$

or, equivalently

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

Note the convention for designating a vector quantity using bold face. This is the convention we follow in the text. In lecture it is difficult to write with chalk in bold face. It is also difficult for you to do so on your homework and exams. In these instances we will use the convention of placing a bar (or twiddle) over or under the letter to indicate it is a vector quantity.

Exercise 2.4

For each of the three force systems shown below, estimate the magnitude of the resultant, of F_1 and F_2. What is the direction of the resultant in each case?
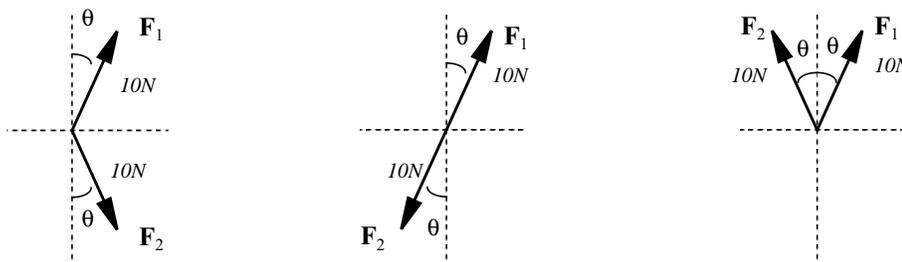
For the force system at the left, the vertical component of the resultant force must be zero since the vertical components of the two forces are equal but one is positive (upward) and the other negative (downward). The symmetry with respect to the horizontal also leads to the result that the horizontal component of the resultant will be the sum of the two (equal) horizontal components of the two

forces. I estimate this sum to be 8 *Newtons* since the angle θ looks to be something less than 30°. The resultant is directed horizontally to the right.

For the system in the middle, the resultant is zero since the two forces are equal but oppositely directed. If we were concerned with the static equilibrium of these three systems, only this system would be in equilibrium.
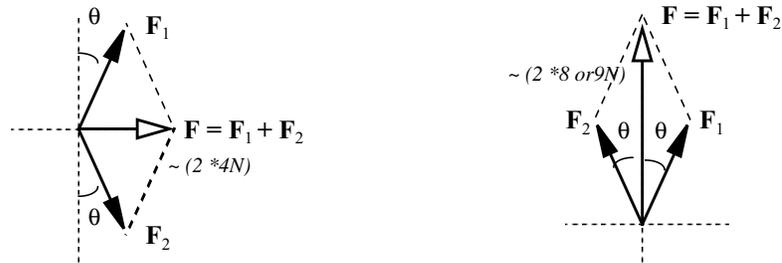

For the system at the far right, with the two forces symmetric with respect to the vertical, we now have that the horizontal component of the resultant force must be zero while the vertical component of the resultant will be the sum of the two (equal) vertical components of the two forces. I estimate this sum to be 16 ~ 18 *Newtons*. The resultant is directed vertically upward as shown above.

Friction Force

Often the greatest challenges in applying the requirements of static equilibrium to useful purpose is isolating the particle (or later the body) and showing on your isolation, your *free-body diagram*, all the forces (and later all the torques as well) that act. These include reaction forces as well as forces applied like the weight due to gravity. Imagining and drawing these forces takes a certain facility in the creative use of the language of Engineering Mechanics, in particular, a facility with the characteristics of different kinds of forces. One of the kinds you will be responsible for reading out of a problem statement and writing into your free-body diagram is the *force due to friction*.

Friction is tricky because sometimes it can be anything it needs to be; it's direction as well as magnitude have a chameleon quality, taking on the colors that best meet the requirements of static equilibrium. But it can only be so big; it's magnitude is limited. And when things begin to move and slide, it's something else again. Friction is even more complicated in that its magnitude depends upon the surface materials which are in contact at the interface you have constructed in your free-body diagram but *not* upon the area in contact. This means that you have to go to a table in a reference book, ask a classmate, or call up a supplier, to obtain an appropriate value for the *coefficient of friction*.

Consider the following illustration of the practical implications of friction and the laws of static equilibrium: I know from experience that my back goes out if I pull on a cable angling up from the ground with a certain force, approximately fifty pounds. Will my back go out if I try to drag the heavy block shown by pulling on the cable *AB*?
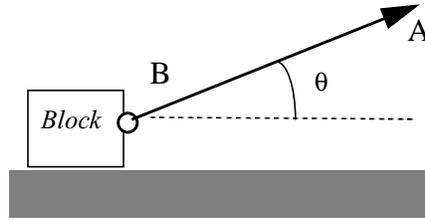
To find the force I must exert, through the cable, onto the block in order to slide it to the right I will isolate the block and apply the laws of static equilibrium. Now this, at first reading, might appear a contradiction: How can the block be sliding to the right and in static equilibrium at the same time?

Two responses are possible:

- If the block is sliding to the right at constant velocity, then the laws of static equilibrium still apply – as when you ascended in an elevator at constant velocity. There is no contradiction.

- If the block does not move, but is *just about to move*, then the laws of static equilibrium still apply and again there is no contradiction. In this case we say we are at the point of *impending motion*; the smallest increment in the force with which I pull on the cable will start the block sliding to the right.

It is the second case that I will analyze, that of *impending motion*. It is this case that will most likely throw my back out.

I first isolate the block as a particle, showing all the applied and reaction forces acting upon it. The weight, **W**, and the force with which I pull the cable *AB*— I will call it $\mathbf{F}_A$— are the applied forces.

The reaction forces include the force of the ground pushing up on the block, **N**, what is called a *normal force*, and the friction force, $\mathbf{F}_f$, acting parallel to the plane of contact of the block with the ground, tending to resist motion to the right, hence acting on the block to the left. For static equilibrium the resultant force on the particle must vanish. Again, this is equivalent to demanding that the sum of the horizontal components and the sum of the vertical components vanish independently.
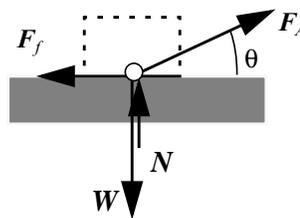
Thus: $F_A \cos\theta - F_f = 0$ and $F_A \sin\theta + N - W = 0$

These are two scalar equations. But look, there are at least three unknowns – F_f, F_A and N and θ. Even if θ is given, say 60°, we are still in a fix since there

remains one more unknown over the number of independent equations available. Now it's not that we can't find a solution; indeed we can find any number of sets of the three unknowns that serve: Just pick a value for any one of the three – F_A, F_f, or N as some fraction of W, say W/2, and use the equilibrium equations to solve for the remaining two. The problem is we can not find a *unique* solution. We say that system is under determined, or indeterminate.

This is where *impending motion* comes to our rescue. We add the condition that we are at the point of impending motion. At impending motion the frictional force is related to the normal force by

$$F_f = \mu \cdot N$$

where μ is called *the coefficient of static friction.*

Note that μ is a *dimensionless* quantity since both the normal force *N* and its associated friction force F_f have the same dimensions, that of force. The particular value of the coefficient of static friction depends upon the materials in contact at the interface where the friction force acts. Another coefficient of friction is defined to cover the case of sliding at constant velocity. It is labeled the coefficient of sliding friction. It too depends upon the character of the materials in contact; it's value is nominally less than μ.

This is the third equation that allows us to estimate the force that will throw my back out. In fact, solving the three equations we find:

$$F_A = \mu \cdot W / (cos\theta + \mu \cdot sin\theta)$$

From this expression I can estimate the weight of the block I might be able to move by pulling on the cable *AB*. For example, if $\theta = 60^o$ and I take $\mu = 0.25$, as an estimate for sliding blocks along the ground, then setting $F_A = 50$ *lb.* — an estimate of the maximum force I can exert without disastrous results — and solving for W, I find from the above

$$W = 143.3 \text{ lb.}$$

The friction force at impending motion is in this case, from the first equation of equilibrium,

$$F_f = F_A cos\theta = 50 \cdot cos\, 60^o = 25 \text{ lb.}$$

If I pull with a force *less than* fifty pounds, say twenty pounds, still at an angle of 60°, on a block weighing 143.3 pounds or more, the block will not budge. The friction force F_f is just what it needs to be to satisfy equilibrium, namely $F_f = 20$ cos $60^o = 10$ *lb*. This is what was meant by the statement " ... can be anything it has to be." The block in this case does not move, nor is it just about to move. As I increase the force with which I pull, say from twenty to fifty pounds, the frictional force increases *proportionally* from ten to twenty-five pounds, at which point the block begins to move and we leave the land of Static Equilibrium. That's okay; I know now what weight block I can expect to be able to drag along the ground without injury. I need go no further.

But wait! What, you say, if I *push* instead of pull the block? Won't pushing be easier on my back? You have a point: I will now analyze the situation given that the *AB* is no longer a cable which I pull but signifies my arms pushing. In this we keep θ equal to 60°.

At first glance, you might be tempted as I was, when I was a youth with a good back but little facility in speaking Engineering Mechanics, to simply change the sign of F_A in the equilibrium equations and let it go at that: Solving would simply change the sign in our final expression for F_A in terms of W. But that will not do.

Friction is trickier: **Friction always acts in a direction resisting the impending motion**. Here is another way it changes its colors to suit the context. No, I can't get away so simply; I must redraw my free-body diagram carefully showing the new directions of the forces F_A *and* F_f. In this I will label the force I apply, F_B

Equilibrium now gives, taking horizontal components to be positive when directed to the right and vertical components positive when directed upward:

$$ -F_B cos\theta + F_f = 0 \quad \text{and} \quad -F_B sin\theta + N - W = 0 $$

These two, again supplemented by the relationship between the friction force and the normal force, namely

$$ F_f = \mu \cdot N $$

now yield

$$ F_B = \mu \cdot W / (cos\theta - \mu \cdot sin\theta) $$

This is the force I must push with in order to just start the block sliding to the left – a state of impending motion. If I push with a force less than this, the block will not budge, the friction force is whatever the first equilibrium equation says it has to be. What force must I push with in order to move a block of weight $W = 143.3$ *lb*? Again, with θ = 60°, the above expression gives $F_B = 126.3$ lb.

Note well the result! I must push with more than twice the force I must pull with in order to move the block! There is no mystery here. The reasons for this are all contained in the equations of equilibrium and the rules we have laid out which govern the magnitude and direction of the force due to friction. It is the latter that adds so much spice to our story. Note, I might go on and construct a story about how pushing down at an angle adds to the normal force of reaction which in turn implies that the frictional force resisting motion at the point of impending motion
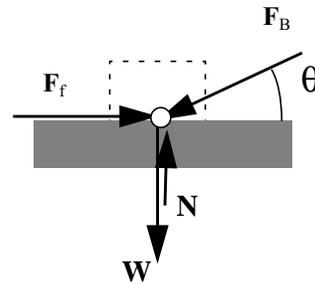
will increase.   The bottom line is that it takes more force to push and start the block moving than to pull and do the same.

Exercise 2.5

Professor X, well known for his lecturing theatrics, has thought of an innovative way to introduce his students to the concept of friction, in particular to the notion of impending motion. His scheme is as follows: He will place a chair upon the top of the table, which is always there at the front of the lecture hall, and ask for a volunteer from the class to mount the table and sit down in the chair. Other volunteers will then be instructed to slowly raise the end of the table. Students in the front row will be asked to estimate the angle φ as it slowly increases and to make a note of the value when the chair, with the student on board, begins to slide down the table surface, now a ramp.

Unfortunately, instead of sliding, the chair tips, the student lurches forward, fractures his right arm in attempting to cushion his fall, gets an A in the course, and sues the University. Re*construct* what Professor *X* was attempting to demonstrate and the probable cause of failure of the demonstration?
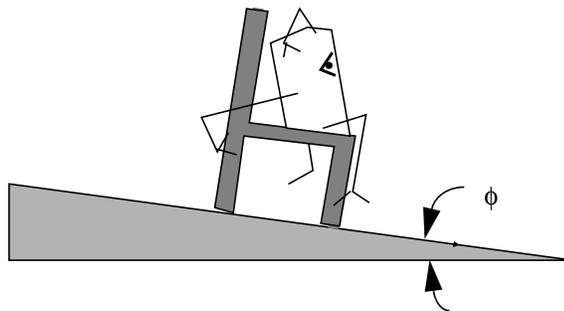
I begin by drawing a free-body diagram, isolating the student and the chair together – all that which will slide down the table top when tipped up – as a particle. The weight W in the figure below is the weight of the student and the chair. The reaction force at the interface of the chair with the table is represented by two perpendicular components, the normal force N and the friction force F_f. We now require that the resultant force on the particle vanish.
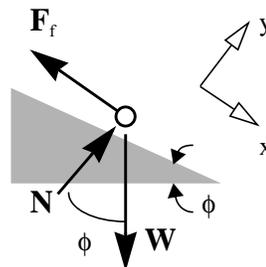
In this, we choose the *x-y* axes shown as a reference frame. We make this choice to simplify our analysis. Only the weight vector W has both *x* and *y* components. Make a mental note of this way of crafting in setting up a problem. It is a bit of nuance of language use that can help you express your thoughts more efficiently than otherwise and yield a rich return, for example, on a quiz when time is precious. Equilibrium in the *x* direction, positive down the plane, then requires

$$-F_f + W \cdot sin\phi = 0$$

while equilibrium in the *y* direction yields

$$N - W \cdot cos\phi = 0$$

I manipulate these to obtain

$$F_f = N\,tan\phi \qquad \text{where} \qquad N = W \cdot cos\phi$$

Now we know from the previous friction problem we analyzed that the friction force can only get so large *relative to the normal force* before motion will ensue. For the problem at hand, once the ratio of friction force to normal force reaches a value equal to the coefficient of static friction, μ, appropriate for the chair's leg tips interfacing with the material of the table's top, motion of the "particle" down the plane will follow. We can state this condition as an *inequality*. The student and chair will not slide down the plane as long as
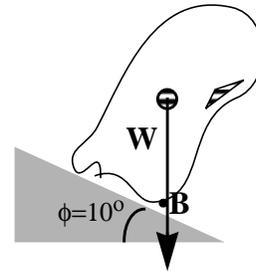
$$F_f = N \cdot tan\phi < \mu \cdot N$$

This immediately yields the conclusion that as long as the angle ϕ is less than a certain value, namely if

$$\boxed{tan\phi < \mu}$$

the particle will not move. Note that, on the basis of this *one-to-one* relationship, we could define the condition of impending motion between two materials in terms of the value of the angle ϕ as easily as in terms of μ. For this reason, ϕ is sometimes called the *friction angle*. For example, if $\mu = 0.25$ then the angle at which the chair and student will begin to slide is $\phi = 14^o$ Note, too, that our result is independent of the weights of the student and the chair. *All* students should begin to slide down the plane at the same angle. This was to be a central point in Professor *X*'s demonstration: He planned to have a variety of students take a slide down the table top. Unfortunately the tallest person in class volunteered to go first.

Why did the demonstration fail?   It   failed because Professor *X* saw a *particle* where he should have envisioned an *extended body*. The figure at the below is an adaptation of a sketch drawn by a student in the front row just at the instant before the student and chair tipped for-ward. Note that the *line of action* of the weight vector of the chair-student combination, which I have added to the student's sketch, passes through the point of contact of the front legs of the chair with the table top, point *B*. Note, too, that the angle ϕ is less than the friction angle, less than the value at which the chair would begin to slide.

In the next instant, as the students charged with lifting the left end of the table did as they were told and raised their end up an inch, the line of action of *W* fell forward of point *B* and the accident ensued.

When is a particle a particle?

The question perhaps is better phrased as "When is a body a particle?" The last exercise brings forcibly home how you can go wrong if you mistakenly read a par-ticle where you ought to imagine something of more substance. We have here a failure in *modeling*.

Modeling is a process that requires you to represent "reality" in the language of Engineering Mechanics, to see in the world (or in the text in front of you) the conceptual ingredients of force, now of torque or moment and how the laws of static equilibrium and subsidiary relations like those that describe the action of a force due to friction, are to apply. It was Professor X's failure to see the tipping moment about point B that led to his, rather the student's, downfall.

Modeling failures are common, like the cold. And there is no easy fix nor med-icine to prescribe that will ensure 100% success in modeling. One thing is essen-tial, at least here at the start: You must draw an isolation, a free-body diagram, as a first, critical step in your response to a problem. That until now, has meant, not just drawing a point on a clean sheet of paper – anyone can do that – but that you imagine all the force vectors acting on the particle and draw them too on your sheet of paper.

This requires some thought. You must imagine; you must take risks; you must conjecture and test out your conjecture. In this you have available the beginnings of a vocabulary and some grammar to help you construct an appropriate isolation of, at least, a particle:

- gravity acts downward;
- friction force acts to resist motion;
- the normal force acts perpendicular to the plane of contact;
- a cable can only sustain a tensile force;

- to every action there is an equal and opposite reaction.

To these notions we can add:

- it doesn't matter where you show a force vector acting along its *line-of-action*;[3]

- you are free to choose the orientation of a reference coordinate system;

- the requirement that the resultant force of *all* the forces acting on an *isolated* particle vanish is equivalent to requiring that the sums of the usually orthogonal, scalar components of all the forces vanish.

Knowing all of this, there remains ample room for error and going astray. Professor *X*'s free-body diagram and analysis of a particle were well done. The failing was in the field of view right at the outset; Modeling a student in a chair on an inclined plane as a particle was wrong from the start.

Now this really makes life difficult since, for some purposes, the chair and student might be successfully modeled as a particle, e.g., if the coefficient of friction is sufficiently small such that I need not worry about the tipping forward, (see problem 2.12), while at other times this will not do. Or consider the block I pull along the ground in section 2.1.3: I successfully modeled the block as a particle there. Note how, at this point, you might now conjecture a scenario in which I could *not* claim success, for example, if the geometry were such that the block would lift off the ground before sliding. Or consider the airplane of Exercise 2.1. If I am interested in the resultant lift force, I can get away with modeling the football-field size machine as a particle; on the other hand, if I were responsible for defining how to set the flaps to maintain a specified attitude of the craft, I would have to take the airplane as an extended body and worry about the distribution of the lift force over the wings and the horizontal stabilizer.

We conclude that the chair and student, indeed all things of the world of Engineering Mechanics, do not appear in the world with labels that say "I am a particle" or "I am not a particle". No, it is *you* who must provide the labels, read the situation, then articulate and compose an abstract representation or *model* that will serve. In short, something might be a particle or it might be an extended body depending upon your interests, what questions you raise, or are raised by others for you to answer.

Now, in most texts whether something is a particle or a body can be easily imputed from the context of the problem. You expect to find only particles in a chapter on "Static Equilibrium of a Particle". On the other hand, if an object is dimensioned, i.e., length, width and height are given on the figure, you can be quite sure that you're meant to see an extended body. This is an usually unstated rule of textbook writing – authors provide all the information required to solve the problem and *no* more. To provide more, or less, than what's required is considered, if not a dirty trick, not in good form. I will often violate this norm. Engi-

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3. This is true as long as we are not concerned with what goes on within the boundary we have drawn enclosing our free body.

neers, in their work, must deal with situations in which there is an excess of information while, at other times, situations in which there is insufficient information and conjecture and estimation is necessary. It's best you learn straight off a bit more about the real world than the traditional text allows.

2.2   Concept of Moment

Force is not enough. You know from your studies in physics of the dynamics of bodies other than particles, that you must speak about their rotation as well as translation through space; about how they twist and turn.

Equilibrium of a Body

We turn, then, to consider what we can say about forces, applied and reactive, when confronted with a body that cannot be seen as a particle but must be taken as having finite dimensions, as an *extended* body. Crucial to our progress will be the concept of *moment* or *torque* which can be interpreted as the *turning effect of a force*.

We start again with a block on the ground. Instead of pushing or pulling, we explore what we can do with a lever. In particular we pose, as did Galileo (who also had a bad back),

Exercise 2.6

Estimate the magnitude of the force I must exert with my foot pressing down at B to just lift the end of the block at A up off the ground?
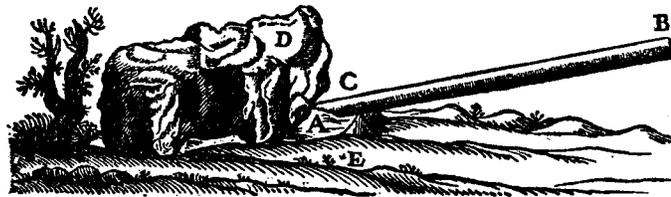
Fig. 15

We isolate the system, this time as an extended body, showing all the applied and reaction forces acting on the system. The applied forces are the weight acting downward along a vertical line of action passing through the *c.g.*, the *center of*

gravity of the block, and the force of my foot acting downward along a line of action through the point *B* at the right end of the lever *AB*.
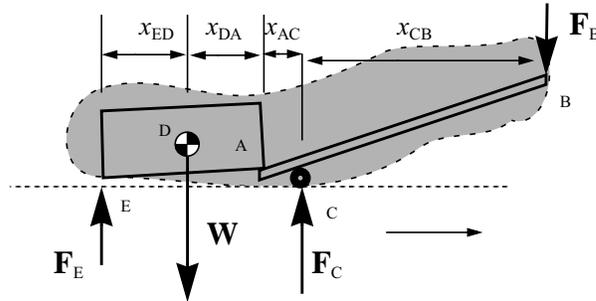
The reaction forces are two: (1) the force of the ground acting up on the left end of the block at *E* and (2), the force of the ground acting up through the pivot at *C* upon the lever *AB*. Our quest is to determine the magnitude of the (vertical) force we must apply at *B* in order to *just lift* the end *A* off the ground. We start by applying our known requirement for static equilibrium – **for a body at rest, or moving with constant velocity, the resultant force acting on the** *isolated body* **must be zero, must vanish**. We have, taking up as positive,

$$F_E - W + F_C - F_B = 0$$

We read this as one (scalar) equation with three unknowns, the applied force F_B and the two reactions F_E and F_C. Clearly we need to say something more. That "more" is contained in the following equilibrium requirement for an extended body – **for a body at rest, or moving with constant velocity, the resultant moment of all forces acting on the** *isolated body* **must be zero, must vanish**. I will find the resultant *moment* or *torque* of all the forces *about the left-most point E*. I will take as positive, a torque which tends to rotate the extended body of block and lever – all that lies within the dotted envelope – clockwise. For example, the moment about point *E* of the reaction force F_C is negative since it tends to rotate the system counter-clockwise about the reference point *E*. Its value is given by $(x_{ED}+x_{DA}+x_{AC})\, F_C$, the product of the force F_C and the perpendicular distance from the point *E* to the *line of action* of the force F_C.

The *resultant moment* of all the forces acting on the isolated system is

$$x_{ED} \cdot W - (x_{ED} + x_{DA} + x_{AC}) \cdot F_C + (x_{ED} + x_{DA} + x_{AC} + x_{CB}) \cdot F_B = 0$$

We may read this equation as a second scalar equation in terms of the three unknown force quantities *if* we take the *x*'s, the distance measures, as known. We might, at this point, estimate the distances: the block length looks to be about one meter.[4] Then, from the figure, estimate the other lengths by measuring their magnitudes relative to the length of the block. I will not do this. Instead, for reasons that will become evident, I will *not* state the block length but simply label it L then figure the x's in terms of L.

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4. A better estimate might be obtained if the reader could identify the shrub at the left of the block. But that's beyond the scope of the course.

My estimates for the lengths are then:

$$x_{ED} = x_{DA} = L/2 \qquad x_{AC} = L/5 \qquad x_{BC} = 7L/5$$

With these, my *equation of moment equilibrium* becomes

$$(L/2) \cdot W - (6L/5) \cdot F_C + (13L/5) \cdot F_B = 0$$

Now make note of one feature. L, the length of the block is a common factor; it may be extracted from each term, then "cancelled out" of the equation. We are left with

$$(W/2) - (6/5) \cdot F_C + (13/5) \cdot F_B = 0$$

Where do we stand now? We have two equations but still three unknowns. We are algebraically speaking "up a creek" if our objective is to find some one, useful measure of the force we must exert at *B* to just lift the block of weight W off the ground at the end A. Again, it's not that we cannot produce a solution for F_B in terms of W; the problem is we can construct many solutions, too many solutions, indeed, an infinite number of solutions. It appears that the problem is indeterminate.

In the next chapter we are going to encounter problems where satisfying the equilibrium requirements, while necessary, is not sufficient to fixing a solution to a problem in Engineering Mechanics. There we will turn and consider another vital phenomenon - the deformation of bodies. At first glance we might conclude that the problem before us now is of this type, is *statically indeterminate*. That is **not** the case. Watch!

I will dissect my extended body, isolating a portion of it, namely the block alone. My *free-body diagram* is as follows:

I have constructed a new force F_A, an *internal force*, which, from the point of view of the block, is the force exerted by the end of the lever at *A* upon the block. Now **this** extended body, **this** subsystem is also in static equilibrium. Hence I can write

$$F_E - W + F_A = 0$$

ensuring *force equilibrium* and $\qquad W \cdot (L/2) - F_A \cdot L = 0 \qquad$ ensuring *moment equilibrium about point E*. The second equation gives us directly $\quad F_A = W/2$

while the first then yields $\quad F_E = F_A = W/2 \quad$ which we might have concluded from the symmetry of our free-body diagram[5]. My next move is to construct yet another isolated body, this time of the lever alone.

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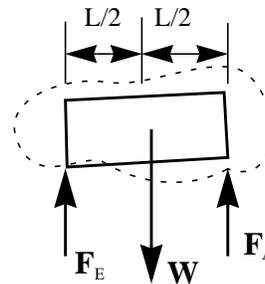
5. Perceiving this symmetry depends upon knowing about the requirement of moment equilibrium of an isolated body so it's a bit unfair to suggest you might have been able to "see" this symmetry without this knowledge.
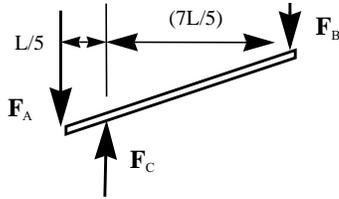
Note that the force acting on the lever due to the block is the *equal and opposite internal reaction F_A* whose magnitude we now know. **This is an essential observation**. Now **this** extended body, **this** subsystem is also in static equilibrium. Hence I can write

$$-W/2 + F_C - F_B = 0$$

ensuring *force equilibrium* and

$$-(L/5) \cdot W/2 + ((7L)/5) \cdot F_B = 0$$

ensuring *moment equilibrium about point C*. Note the repetition in language here and with that of the analysis of the free-body diagram of the block alone. Indeed, once we have constructed the abstract representation, the free-body diagram, the subject matter becomes somewhat boring and repetitive, machine-like. From the second equation, that of moment equilibrium, I find $F_B = W/14$ which is a significant *mechanical advantage*.

- Observe that if I had *taken moments* about point *A* shown in the free-body diagram of the lever, I would have obtained a *different* equation expressing moment equilibrium namely, $-(L/5) \cdot F_C + ((8L)/5) \cdot F_B = 0$

- but I would obtain the same result, the same answer. However, I would have to make use of the equation of force equilibrium together with this last equation of moment equilibrium to get to the answer. This feature of this particular problem may be generalized, to wit: **It doesn't matter what point in space you choose as a reference point when you construct an equation of moment equilibrium.** This is powerful knowledge that may dramatically increase your productivity for often, by judicious choice of a reference point, you can simplify your analysis.

- Observe too that in all three isolations the forces were read as planar and parallel, that is their lines of action were drawn in a single plane and parallel to the vertical. In each of the three cases, for each isolation, we wrote out two independent, scalar equations; one expressed force equilibrium in the vertical direction, the other moment equilibrium about some reference point. Now I could have, instead of force equilibrium, applied moment equilibrium again, about some *other* reference point. For example, for the lever, the last isolation diagram constructed, if I take moment equilibrium about the left end, this, together with the consequence of moment equilibrium about point *C*, namely $F_B = W/14$ produces the same result for the reaction at *C*. Check it out.

- Observe that at a point early on in our analysis we might have concluded that we had insufficient information to do the problem. But, by breaking down the problem into two other problems we found our way to a solution.

- Observe, finally, that, after having analyzed the block as an isolated sub-system and obtained the reaction force, F_E, we could have gone directly back to the original three equations of equilibrium and solved for the remaining two unknowns, F_C and F_B. Once again we note that there are alternative paths to a solution. Some paths are more direct than others; some are more enlightening than others, but they all should lead to the *same* solution if the question is well-posed.

We begin to see now the more subtle aspects of applying the requirements of static equilibrium to useful purpose: Effective use of this new language will require us to make choices – choices of reference points for taking moments, selection of subsystems to analyze when one free-body diagram won't yield all we need to know – and requires a familiarity with different renderings of force and moment equilibrium. There is no unique, cook-book, 100% sure method to solving problems, even statics problems, in Engineering Mechanics.

Different Kinds of Systems of Forces

The requirements of force equilibrium and moment equilibrium are two *vector* equations. We can write them as:

$$\sum \mathbf{F_i} = 0 \qquad \sum \mathbf{M_i} = 0$$

In these two **vector** equations, the summation is to be carried out over all forces and moments acting on an isolated body— the **i** ranges over one to N forces say. We can interpret the first as the resultant of all externally applied forces and the second as the resultant moment of all the forces (and other, *concentrated moments or couples,* yet to be defined) acting on the body.

The resultant force on a particle or body and the resultant moment are both vector quantities; each has a magnitude and a direction which must be specified to fully know the nature of the beast. Each vector resultant has three (3) scalar components in three-dimensional space so each vector equation is equivalent to three *independent* scalar equations. From this we conclude:

- **There are at most six (6) independent scalar equations available (which must be satisfied) to ensure static equilibrium of an isolated body. For a particle, there are at most three (3) independent scalar equations available.**

If you look back over the exercises we have worked in the preceding sections of this text you will note: – estimating the lift force on an aircraft required citing a single scalar equation; in pulling and pushing the block along the ground, with the block taken as a particle, we made use of two scalar equations; so too, our analysis of a particle sliding down a plane required the use of two scalar equations of equilibrium. Nowhere did we need three scalar equations of equilibrium. The reason? All force vectors in each of these particle problems lay in the plane of the page hence each had but two scalar components, two x,y or, in some cases, horizontal, vertical components. Likewise the resultant force shows but two scalar compo-

nents. Force equilibrium is equivalent to setting the sum of the x components and the sum of the y components to zero. Alternatively we could say that force equilibrium in the direction perpendicular to the plane of the page is *identically satisfied*; 0 = 0; since there are no components in this direction.

In our analysis of an extended body, the block with lever applied, we had six scalar equations available, at most. Yet in each of the three isolations we constructed we wrote but two independent scalar equations and that was sufficient to our purpose. How do we explain our success; what about the other four scalar equations? They must be satisfied too.

First note that again, all force vectors lie in the plane of the page. Not only that, but their lines of action are all parallel, parallel to the vertical. Hence force equilibrium in all but the vertical direction is satisfied. That takes care of two of the four.

Second, since the force vectors all lie in a single plane, they can only produce a turning effect, a torque or a moment, about an axis perpendicular to that plane. Thus moments about the axes lying *in the plane*, the x,y axes, will be identically zero. That takes care of the remaining two scalar equations not used.

From all of these observations we can boldly state:

- **If the lines of action of all forces acting on a *particle* lie in a common plane, there are at most two independent, scalar, equilibrium equations available.**

- **If the lines of action of all forces acting on an *extended body* are all parallel and lie in a common plane there are at most two independent, scalar, equilibrium equations available.**

- **If the lines of action of all forces acting on an *extended body* all lie in a common plane there are at most three independent, scalar, equilibrium equations available.**

Note well, however, that these are statements about the *maximum number* of independent equations available to us in particular contexts. They do *not* say that so many must derive from moment equilibrium and so many from force equilibrium. We have seen how, in the analysis of the lever used in lifting the end of Galileo's block up off the ground, we were able to apply moment equilibrium twice to obtain a different looking, but equivalent, system of two linearly independent equations – different from the two obtained applying moment equilibrium once together with force equilibrium in the vertical direction.

Similarly for the block sliding, as a particle, down the plane we might have oriented our x,y axes along the horizontal and vertical in which case the two scalar equations of force equilibrium would appear, for the horizontal direction,

$$-F_f \cdot cos\phi + N \cdot sin\phi \ = \ 0$$

and, for the vertical direction,

$$N \cdot cos\phi + F_f \cdot sin\phi - W \ = \ 0$$
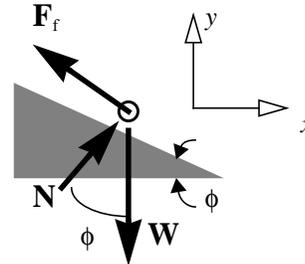
These two are equivalent to the two force equilibrium equations we previously derived; they yield the same solution; they are synonymous; the two sets have identical meaning.

The requirements of force and moment equilibrium have further implications for particular systems of forces and moments. We use them to define *two force members* and *three-force members*.

Exercise 2.7

What do you need to know in order to determine the reaction forces at points A and B of the structure shown below?
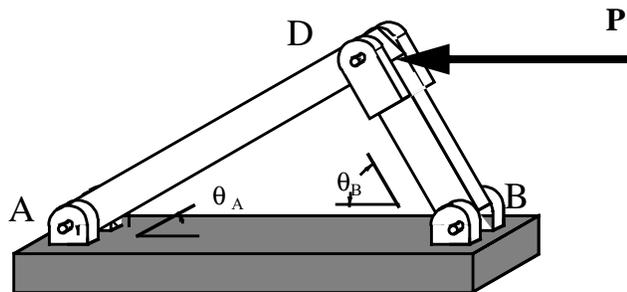
In that this problem, at least with regard to its geometry, looks like the block hanging from two cables of Exercise 2.2, we might begin by mimicking what we discovered we needed to know there:

- You need to know the force applied *P*, just as we needed to know the weight of the block in Exercise 2.2.

- You need to know the angles θ_A and θ_B

The other item needed-to-know doesn't apply since members *AD* and *BD* are not cables. To go further we will now try to solve the problem. If we are able to solve the problem we surely then should be able to say what we needed to know.

On the other hand, it is entirely possible that a neophyte might be able to solve a problem and yet not be able to articulate what they needed to know to get to that point. This is called muddling through and ought not to be condoned as evidence of competence.

I first isolate the system, cutting the structure as a whole away from its moorings at *A* and *B*.

Letting A_x, A_y, B_x and B_y be the horizontal and vertical components of the reaction forces at *A* and *B*, force equilibrium will be assured if

$$A_x + B_x - P = 0$$

and

$$A_y + B_y = 0$$

Our third equilibrium equation for this planar system of forces acting on an extended body may be obtained by *taking moments about point A*[6].

We have

$$(L_{AD} \cdot cos\,\theta_A + L_{BD} \cdot cos\,\theta_B) \cdot B_y - P \cdot L_{AD} \cdot sin\,\theta_A = 0$$

In this, I have taken clockwise as positive. Observe that the two lengths that appear in this equation are related by

$$L_{AD} \cdot sin\,\theta_A = L_{BD} \cdot sin\,\theta_B$$

At first glance it would appear that I must know one of these lengths in order to solve the problem. This is *not* the case since the two lengths are *linearly related* and, after eliminating one of them from moment equilibrium, the other will be a factor common to all terms in this equation, hence, will not appear in the final solution for the reaction force components at *A* and *B*.

Proceeding in this way I can solve the equation expressing moment equilibrium and obtain B_y in terms of the applied load P.

$$B_y = -P \cdot [\,sin\,\theta_B \cdot sin\,\theta_B\,] / [\,sin\,(\theta_A + \theta_B)\,]$$

Force equilibrium in the *y* direction, the second equilibrium equation, then gives

$$A_y = -B_y = P \cdot [\,sin\,\theta_B \cdot sin\,\theta_B\,] / [\,sin\,(\theta_A + \theta_B)\,]$$

So far so good.... but then again, that's as far as we can go; we are truly up a creek. There is no way we can find unique expressions for A_x and B_x in terms of P, θ_A, and θ_B without further information to supplement the first equilibrium equation. But watch!

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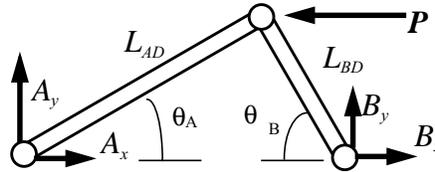
6. This is a bit of common dialect, equivalent to stating that the sum of the moments of all forces with respect to A will be set to zero in what follows. Note too how, in choosing A as a reference point for moments, the components of the reaction force at A do not enter into the equation of moment equilibrium.

I isolate a subsystem, member *AD*. The circles at the ends of the member are to be read as *frictionless pins*. That means they can support *no* moment or torque locally, at the end points.

These circles act like a ring of ball bearings, offering no resistance to rotation to the member about either end. This implies that only a force can act at the ends. These might appear as at the right. Now we apply equilibrium to this isolated, extended body $\mathbf{F}_A + \mathbf{F}_D = 0$.

This implies that the two internal forces must be equal and opposite. Our isolation diagram might then appear as on the left, below
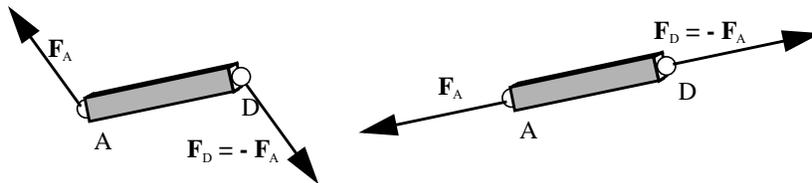
Force equilibrium is now satisfied but surely moment equilibrium is not. The two forces as shown produce an unbalanced *couple*. For this to vanish we require that the two lines of action of the two forces be coincident, co-linear and our isolation diagram must appear as on the right.

We can go no further. Yet we have a very important statement to make:

> • **For a two force member, an isolated body with but two forces acting upon it, the two forces must be equal, opposite, and colinear.**

For the straight member *AD*, this means that the lines of action of the forces acting at the ends of the member lie along the member. We say that *the two-force (straight) member is in tension or compression.*

Knowing that the force at *A* acting on the member *AD* acts along the member, we now know that the direction of the reaction force at *A* because the reaction force at *A is* the same force. This follows from isolating the support at *A* .

Here we show the reaction force at *A* with its components A_x and A_y now drawn such that $A_y / A_x = tan\,\theta_A$
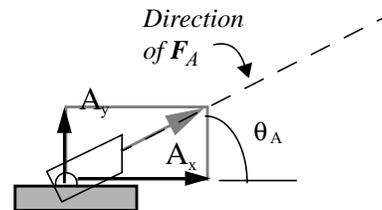
This last equation, together with the first equilibrium equation and our previously obtained expression for A_y, enables us to solve for both A_x and B_x. And that completes the exercise.

At this point, the reader may feel cheated. Why not go on and find the reaction forces at the points *A* and *B*? No. I have solved the *need to know problem* as it was posed. I have constructed a well posed problem; I can farm it out to some subordinate to carry through the solution, confident that anyone with a basic knowledge of algebra can take it from here. The "heavy lifting" is over. So too, you must learn to have confidence in your ability to set-up a well posed problem and to delegate the responsibility for crank-turning to others, even to computational machinery. *However*, you, of course, remain responsible for evaluating the worth of what comes back to you.

In conclusion, in addition to knowing the load P and the angles θ_A and θ_B. in order to solve the problem we need to know:

- How to read a circle as a frictionless pin and what we mean by the phrase *two-force member.*

- Also, again, how to experiment with isolations and equilibrium considerations of pieces of a system.

Some final observations:

- Say this structure was the work of some baroque architect or industrial designer who insisted that member *BD* have the form shown immediately below. While there may be legitimate reasons for installing a curved member connecting *B* to *D*, — e.g., you need to provide adequate clearance for what is inside the structure, this creates no problem. The reactions at *A* and B (see below) will be the same as before (assuming frictionless pins). What *will* change is the mode in which the, baroque, now curved, member *BD* carries the load. It is no longer just in compression. It is subject to *bending*, a topic we consider in a section below[7].
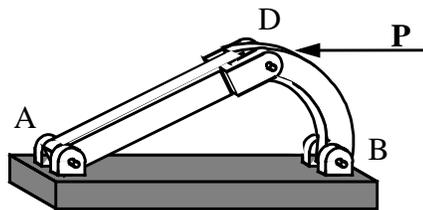
- Returning to our original structure with straight members, t he same sort of analysis of an isolation of member *BD* alone would lead us to conclude that the direction of the reaction at *B* must be up along the member *BD*. Is

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7. Independent of such motives, some will argue that "form follows function"; the baroque for baroque's sake is not only functionally frivolous but ugly.
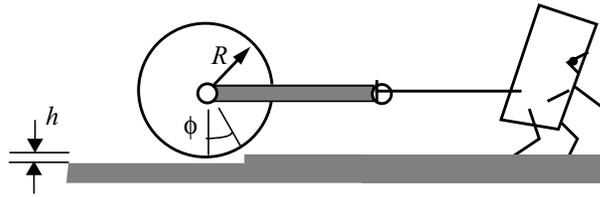
this the case? That is, do the values obtained for B_x and B_y obtained above satisfy the relationship $\tan\theta_B = -B_y/B_x$? Check it out!

Exercise 2.8

Show that the force F required to just start the lawn roller, of radius R and weight W, moving up over the ledge of height h is given by

$$F/W = \tan\phi \qquad \text{where} \qquad \cos\phi = 1 - (h/R).$$

We, as always, start by isolating the system – the roller – showing all the forces

acting upon it. These include the weight acting downward through the center of the roller; the horizontal force applied along the handle by the child laborer (note the frictionless pin fastening the handle to the roller); and the components of the reaction force at the bump, shown as N_x and N_y.
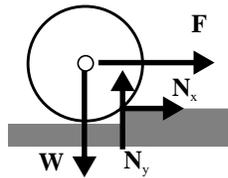
Note the implications of the phrase *just start...* This is to be read as meaning there is no, or insignificant, contact with the ground at any other point than at the bump. Hence the reaction of the ground upon the roller acts at the bump alone.[8] Force equilibrium gives

$$N_y - W = 0 \qquad \text{and} \qquad N_x + F = 0$$

Moment equilibrium will give us a third equation sufficient to determine the three unknowns N_x, N_y and, the really important one, F. I take moments about the center of the roller and immediately observe that the resultant of N_x and N_y must pass through the center of the roller since only it produces a moment about our chosen reference point. That is, the orientation of the reaction force shown below on the left violates moment equilibrium. The reaction force must be directed as shown in the middle figure.

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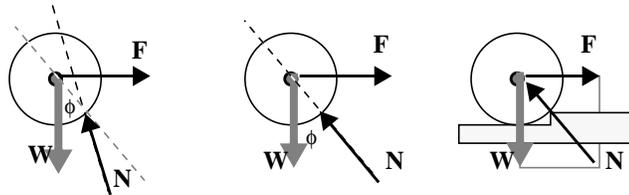
8. It is impossible give you a cookbook rule on how to read phrases like *just start.* Effective working knowledge comes with the exercising of the language. Furthermore I can give you no assurance that an author of another text, will use the exact same phrase to indicate this condition.

Furthermore, we require the three forces to sum to zero to satisfy force equilibrium, so: **F + W + N = 0.**

This is shown graphically in the figure at the far right above. From this we obtain what we were asked to show. (A bit of analytical geometry leads to both relationships). Now from this we can state another rule, namely

> • **For an isolated system subject to but three forces, a** *three force system***, the three forces must be concurrent. That is their lines of action must all run together and intersect at a common point.**

Note the priority of the equilibrium requirements in fixing a solution. We might have begun worrying about the frictional force, or something akin to a frictional force, acting at the bump, resisting motion. But our analysis says there is *no* frictional force! The reaction is perpendicular to the plane of contact; the latter is tangent to the surface at the bump and perpendicular to the radius. How can this be? Nothing was said about "assume friction can be neglected" or "this urchin is pulling the roller up over a bump in the ice on the pond" No, because nothing need be said! Moment equilibrium insists that the reaction force be directed as shown. Moment equilibrium has top priority. The Platonists win again; it's mostly, if not all, in your mind.

Resultant Force and Moment

For static equilibrium of an isolated body, the resultant force and the resultant moment acting on the body must vanish. These are vector sums.

Often it is useful to speak of the resultant force and moment of some **subset** of forces (and moments). For example, a moment vector can be spoken of as the resultant of its, at most, three scalar components. In the figure, the moment or torque about the inclined axis *AA'* is the resultant or vector sum of the three vectors $\mathbf{M}_x$, $\mathbf{M}_y$ and $\mathbf{M}_z$ each of which can be written as the product of a scalar magnitude and a unit vector directed along the appropriate coordinate axis. That is, we can write.
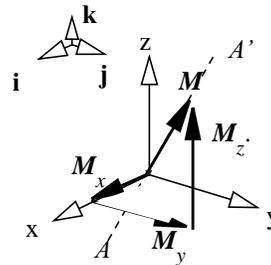
$$\mathbf{M} = \mathbf{M}_x + \mathbf{M}_y + \mathbf{M}_z = M_x \cdot \mathbf{i} + M_y \cdot \mathbf{j} + M_z \cdot \mathbf{k}$$

A moment in itself can be spoken of as the resultant of a force but, while this phrasing is formally correct, it is rarely used. Instead we speak of the *moment at a point due to a force* or the *moment about a point of the force....* as in "the moment, about the point *O* of the force $\mathbf{F}_A$ is given by the product of (1) the perpendicular distance from the point *O* to the line of action of the force and (2) the magnitude of the force. Its direction is given by the *right-hand rule*.

The *right-hand rule* is one of those oddities in science and engineering. It is often stated as

- **the direction of the moment of a force about a point is the same as the direction of advance of a right handed screw when the screw is oriented perpendicular to the plane defined by the line of action of the force and the reference point for the moment.**

This is a mouthful but it works. It meets the need to associate a direction with the turning effect of a force.

We can seemingly avoid this kind of talk by defining the moment as the vector cross product of a position vector from the reference point to *any point* on the line of action of the force. But we are just passing the buck; we still must resort to this same way of speaking in order to define the direction of the vector cross product.

It is worth going through the general definition of moment as a vector cross product[9]. Some useful techniques for calculating moments that avoid the need to find a perpendicular distance become evident

The magnitude of the vector cross product, that is he magnitude of the moment of the force about the point *O* above is
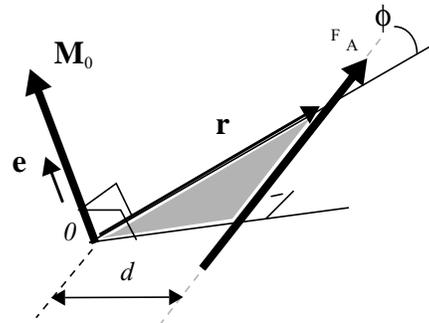
$$\left| \mathbf{r} \times \mathbf{F}_A \right| = \left| \mathbf{M}_O \right| = \left| \mathbf{r} \right| \cdot \left| \mathbf{F}_A \right| \cdot \sin\phi$$

or, striking the bold face to indicate scalar magnitude alone $M_O = r \cdot F_A \cdot \sin\phi$

This is in essence the definition of the vector cross product. The direction of the moment is indicated on the figure by the unit vector **e**. Note: e is commonly used to represent a unit vector. Its heritage is German; *eine* is one.

Note that we recover the more specialized definition of the magnitude of the moment, that which speaks of perpendicular distance from the reference point to the line of action of the force, by writing

$$d = r \cdot \sin\phi \qquad \text{so that} \qquad \mathrm{M}_O = d \cdot F_A$$

Observe that I could have interpreted the magnitude of the moment as the product of *r* and the component of the force perpendicular to the position vector, namely F*sin*φ.

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9. *Vector* and *cross* are redundant when used to describe "product". We ought to speak only of *the vector product* or *the cross product*. Similarly, the *dot product* of two vectors can be called *the scalar product*. It would be again redundant to speak of *the scalar, dot product*. We speak redundantly here in order to emphases that the outcome of a cross product is a vector

In evaluating the cross product, you must take care to define the angle ϕ as the included angle between the position vector and the force vector when the two are placed "tail-to-tail". ϕ is the angle swept out when you swing the position vector around to align with the force vector, moving according to the right hand rule.

The payoff of using the cross product to evaluate the moment of a force with respect to a point is that you can choose a position vector from the point to *any point on the line of action* of the force. In particular, if you have available the scalar components of some position vector and the scalar components of the force, the calculation of both the magnitude and direction of the moment is a machine-like operation.

Exercise 2.9

Show that the moment about point A due to the tension in the cable DB is given by

$$\mathbf{M} = (-0.456 \cdot \mathbf{i} + 0.570 \cdot \mathbf{k}) \cdot L \cdot F_D$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are, as usual, three unit vectors directed along the three axes *x, y, z*.
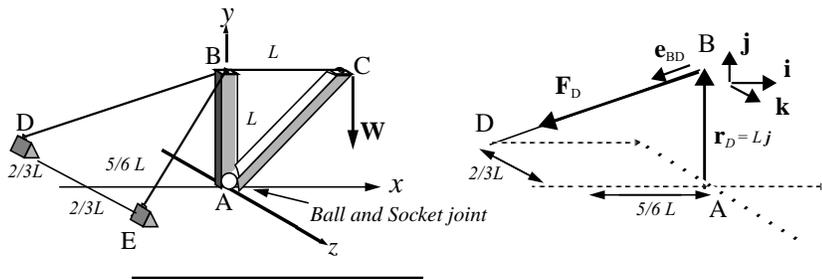
The approach is reductive and mechanical. I will use the vector machinery of the cross product, first expressing a position vector and the force in terms of their *x,y,z* components using the scalar magnitude/unit vector mode of representation for each. For the position vector I write, enjoying the freedom I have to select any point on the line of action of the force as the head of the vector:

$$\mathbf{r}_D = L \cdot \mathbf{j}$$

Simple enough! But now matters become more complex. I write the tension in the cable as the product of its magnitude, F_D and a unit vector directed along its line of action, along the line running from *B* to *D*. Constructing the unit vector

requires the application of the pythagorean theorem and other bits of analytical geometry. From the Pythagorean Theorem, the length of the cable *BD* is

$$L_{BD} = [(5/6)^2 + (1)^2 + (2/3)^2]^{1/2} \cdot L$$

The components of the unit vector, $\mathbf{e}_{BD}$, from *B* to *D* are then
- $(5/6)L/L_{BD}$ along x, $-1L/L_{BD}$ along y, and $-(2/3)L/L_{BD}$ along z. So:

$$\mathbf{e}_{BD} = -(5/6)(L/L_{BD})\mathbf{i} - 1(L/L_{BD}) \cdot \mathbf{j} - (2/3)(L/L_{BD}) \cdot \mathbf{k}$$

or, with L_{BD} defined above and calculating to three significant figures[10],

$$\mathbf{e}_{BD} = -0.570 \cdot \mathbf{i} - 0.684 \cdot \mathbf{j} - 0.456 \cdot \mathbf{k}$$

The moment of the tension in the cable *BD* about *A* is then obtained from the cross product:

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}_D = L \cdot \mathbf{j} \times (-0.570 \cdot \mathbf{i} - 0.648 \cdot \mathbf{j} - 0.456 \cdot \mathbf{k}) \cdot F_D$$

Our cross product is now a collection of three cross products. The machinery for doing these includes

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$

and recognition that the cross product is sensitive to the order of the two vectors, for example $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, and that the vector product of a unit vector with itself is zero. Cranking through, we obtain

$$\mathbf{M} = (-0.456 \cdot \mathbf{i} + 0.570 \cdot \mathbf{k}) \cdot L \cdot F_D$$

- Observe that the moment has a component along the *x* axis (negatively directed), another along the *z* axis but not along the *y* axis; the force produces no turning effect about the vertical axis. Indeed, the force intersects the *y* axis.

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10. Three significant figures are generally sufficient for most engineering work. You are responsible for rounding off the 8, 16 or 20 numbers contemporary spread sheets and other calculating machinery generates. You must learn how to bite off what you can't possible chew.

- Observe that you can deduce from this result the perpendicular distance from *A* to the line of action of the force by recasting the vector term as a unit vector. The factor introduced to accomplish this when associated with the length *L* then defines the perpendicular distance. That is, we write, noting that $[.456^2 + .570^2]^{1/2} = .730$
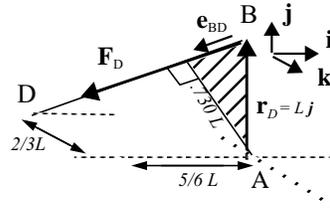
$$\mathbf{M} = (-0.627 \cdot \mathbf{i} + 0.781 \cdot \mathbf{k})(0.730 \cdot L) \cdot F_D$$

and so the perpendicular distance from *A* to the line of action must be 0.730L.

- Finally, observe that I could have constructed a position vector from *A* to *D*, namely

$$\mathbf{r} = -(5/6) \cdot L \cdot \mathbf{i} - (2/3) \cdot L \cdot \mathbf{k}$$

and used this to compute the moment. (See problem 2.8)

Couple

There exists one particularly important resultant moment, that due to two equal and opposite forces – so particular and important that we give it its own name – *couple*. To see why this moment merits singling out, consider the resultant moment about the point O of the two equal and opposite forces $\mathbf{F}_A$ and $\mathbf{F}_B$.
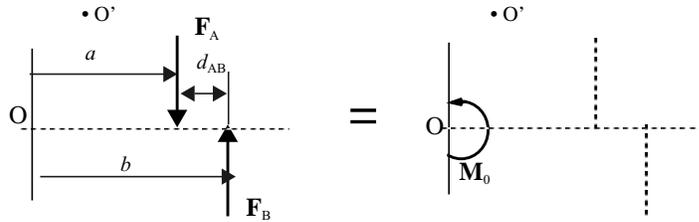
We can write the scalar magnitude of the moment, taking counter clockwise as positive,

$$M_O = -a \cdot F_A + b \cdot F_B = (b - a) \cdot F_A$$

where we have explicitly made use of the fact that the forces are equal and opposite by taking $\mathbf{F}_B = -\mathbf{F}_A$. Note that this equation is written only in terms of the scalar magnitudes of the forces and the moment. The figures on the left and right show two *equivalent systems of forces*.

Note that (*b-a*) is just the (perpendicular) distance, d_{AB} between the two parallel lines of action of the two forces, so if I were to ask "what is the moment of the two forces about some other point?" say O', my response would be the same. In fact

- **the moment due to two equal and opposite forces,** *a couple* **is invariant with respect to choice of reference point; its value is given by the product of the perpendicular distance between the two lines of action of the two forces and the value of the force.** The resultant force due to the two equal and opposite forces is zero.

Exercise 2.10

Show that, for equilibrium of the cantilever beam loaded with two equal and oppositely directed forces as shown, the reaction at the wall is a couple of magnitude Fd and having a direction "out of the paper" or counterclockwise.
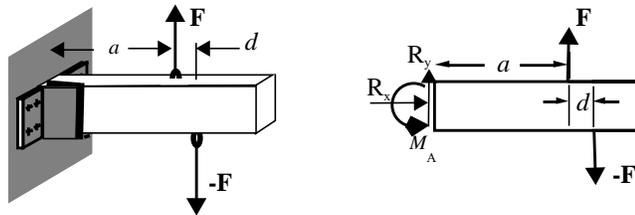
We isolate the cantilever beam, showing all the applied and reaction forces (and couples) acting. This is shown at the right. Note the symbol, the circular arrow. It represents the unknown couple or moment *acting at the point A*, assumed here to be acting counterclockwise. Force equilibrium requires

$$R_x = 0; \qquad R_y + F - F = 0$$

These immediately lead to the conclusion that the reaction force at *A* is zero. R_y as well as R_x are zero.

Moment equilibrium *about the point A*[11] requires

$$M_A - F \cdot d = 0 \qquad \text{so} \qquad M_A = F \cdot d$$

We say *The reaction at A is the* **couple** *Fd*.

_____

11. Note the difference between the two phrases *moment acting at the point or about the point* and *moment* **equilibrium** *about the point*. We say there is a moment (couple) acting *at* the point and then, in the next breath, say that the resultant moment *about* the point must vanish for equilibrium to be satisfied. There is nothing inconsistent here. It is essential that you understand the difference in these two expressions.

The exercise above might strike you as overly abstract and useless, almost a tautology, as if we proved nothing of worth. Don't be fooled. The purpose of the exercise is not algebraic analysis but rather to move you to accept the existence of *couple* as an entity in itself, a thing that is as real as *force*. You may not sense a couple the way you feel a force but in Engineering Mechanics the former is just as lively and substantial a concept as the latter.

Note too how if the beam had been clamped to a wall at its other end, at the right rather than at the left, the reaction at the wall would still be a *couple* of magnitude *Fd* directed counterclockwise.

Exercise 2.11

What if the beam of the last exercise is supported at the wall by two frictionless pins A,B? What can you say about the reaction forces at the pins?
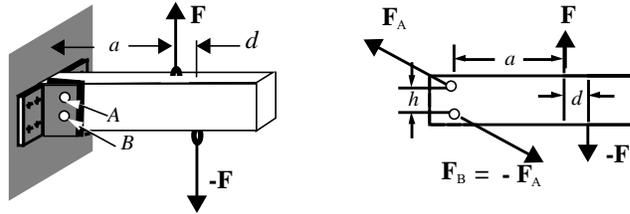
We can say that the two forces acting on the pins, say $\mathbf{F}_A$, $\mathbf{F}_B$, are equal and opposite. We can say that they are equivalent to a *couple* of magnitude Fd and of direction counter clockwise. All of this follows from both force and moment equilibrium requirements applied to the beam when isolated from the wall to which it is pinned. We can say nothing more.

Well, that's not quite the case: While we cannot say what their magnitude is, we can say that they are *at least* as big as the product $F(d/h)$. That is $F_A = F_B \geq F \cdot (d/h)$ because h is the maximum possible perpendicular distance that can be drawn between any two parallel lines of action drawn through A and B.

The couple *Fd* in these few exercises is an *equivalent system* – equivalent to that of two equal and opposite forces separated a distance *d*. In fact, the notion of *equivalent force system* is closely wrapped up with the requirements of force and of moment equilibrium. We can replace one equivalent force system by another in our equilibrium deliberations and the results of our analysis will be the same because

- **equivalent systems have the same resultant force and the same resultant moment about any, arbitrarily chosen point.**

Furthermore, now that we allow a couple vector to be a thing in itself:

- **every system of any number of force vectors and couple vectors can be represented by an equivalent system consisting of a *single* force vector together with a *single* couple vector.**

For example, the three systems shown below are all equivalent
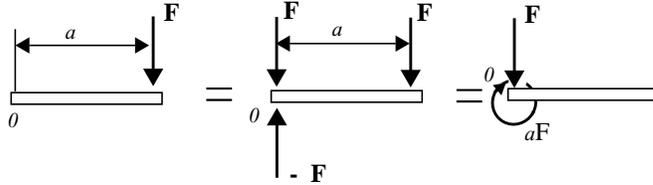
Exercise 2.12

Show that for the load W, uniformly distributed over the span L, (a) an equivalent system is a force vector of magnitude W directed downward acting at the center of the span and no couple; (b) another is a force vector of magnitude W directed downward acting at the left end of the span and a couple of magnitude W(L/2) directed clockwise; another is.....
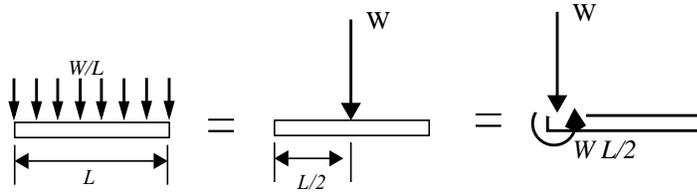
For parts (a) and (b) the resultant force is clearly the total load *W*. In fact, regardless of how the load is distributed over the span, the resultant, equivalent force, will have magnitude *W* and be directed vertically downward. The location of its line of action is a matter of choice. Different choices, however imply different moments or couples — the other ingredient of our equivalent system.

Taking midspan as our reference point, (a), we see that the load, if uniformly distributed, on the right will produce a moment clockwise about center span, that on the left will produce a moment *counter* clockwise about center span. The magnitudes of the two moments will be equal, hence the resultant moment of the uniformly distributed load with reference to center span is zero.

Taking the left end of the span as our reference point, (b), we can make use of the equivalence displayed in the preceding figure on the preceding page where now *a* is read as the distance *L*/2 between the vertical load *W* acting downward at center span and the left end of the span.

Exercise 2.13

Show that a radially directed force per unit length, α, *with units newtons/ meter, uniformly distributed around the circumference of the half circle is equivalent to a single force vector of magnitude 2*α *R, where R is the radius of the half circle, and no couple.*
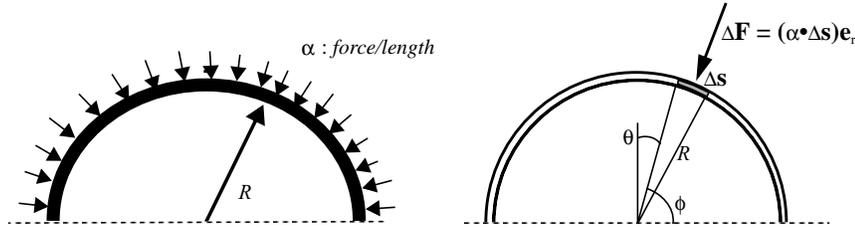
We integrate. We sum up vertical components of differential elements of the distributed force around the circumference.

A differential element of force, $\Delta \mathbf{F}$, acting on the differential length of circumference Δs is where the unit vector $\mathbf{e}_r$ is taken positive inward. The vertical

$$\Delta \mathbf{F} = (\alpha \cdot \Delta s) \cdot \mathbf{e}_r$$

component of this is given by $\alpha \Delta s \cos\theta$ which, noting that $\theta = (\pi/2) - \phi$, can be written $\alpha \sin\phi \Delta s$. Now Δs is just $R\Delta\phi$ so we can write

$$F_{down} = \int_0^\pi \alpha \sin\phi (Rd\phi) = 2\alpha R$$

where the α and *R* are constants.

Design Exercise 2.1

The exercise set out below and the others like it to follow are not complete. It is a design exercise and, as such, differs from the problems you have been assigned up to now. It, and the other design exercises to follow, is different in that there is no single right answer. Although these exercises are keyed to specific single-answer problems in the text and are made to emphasize the fundamental concepts and principles of the subject, they are open-ended. The responses you construct will depend upon how you, your classmates, and your recitation instructor flesh out the task.

In effect, we want you to take responsibility in part for defining the problem, for deciding which constraints and specifications are critical, and setting the context for evaluating possible "solutions". Design is the essence of engineering and the act of design includes formulating problems as much as solving them, negotiating constraints as well as making sure your solutions respect them, and teamwork as well as individual competence.

Hospital-bed Wheel Size
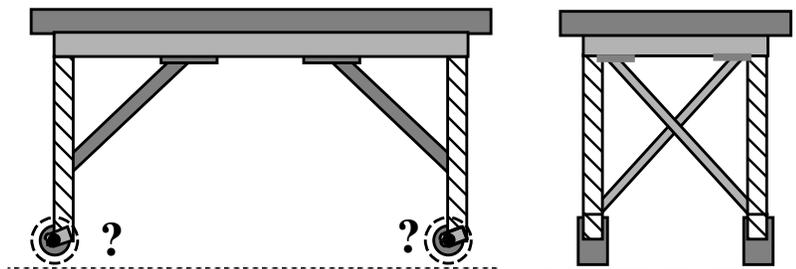
Your task is to do a first-cut analysis in support of the design of a new, lightweight, mobile hospital bed. You know that the bed will be used to transport patients indoors on caster type wheels over relatively smooth terrain but there will be some small obstacles and bumps it must traverse without discomfort to the patient. A single attendant should be able to push the bed to its destination. Develop a rationale for fixing the size of the wheels and use it to determine a range of possible diameters.

Design Exercise 2.2

Your boss wants to diversify. The market for portable stanchions for volleyball nets has diminished over the years with the introduction of more and more cable television stations and the opening of the information highway. People are spending most of their time just lying around, scanning channels. He figures there will be a growing demand for hammocks and the stanchions to support them.

The stanchions are to be portable. He wants them to be able to support the hammock's occupant (occupants?) without fastening the stanchion to the ground. He envisions the ground to be a level surface with most users indoors. The area, the *footprint*, of the hammock and stanchions is limited and you are to assume that there are no walls to tie any supports to.

Your job is to identify the most important design parameters to ensure that the free standing stanchions hold the hammock in the desired form - indicated below. In this you want to construct estimates of the required weight of the stanchions, coefficients of friction, and explore limits on, and ranges of, pertinent dimensions
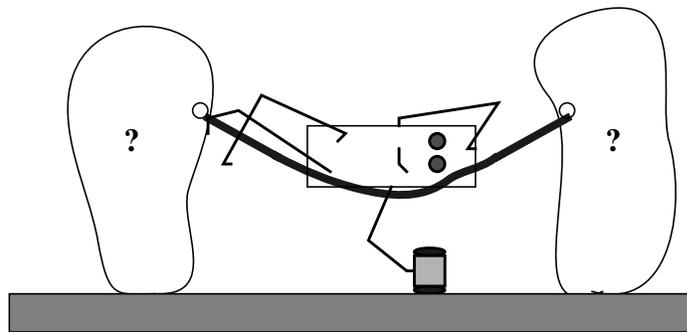
Top vue -- System footprint

of the hammock, the stanchions, as well as the height of attachment and distance between them

2.3   Problems - Static Equilibrium

2.1   *Estimate* the weight of a compact automobile; the weight of a fully loaded trailer truck.

2.2   *Estimate* the weight of a paper clip; the total weight borne by a book shelf one meter long, fully packed with books; the weight of the earth's moon.

2.3   *Estimate* the angle of a hill upon which you can safely park your car under dry road conditions; under icy conditions.

2.4   *Estimate* the maximum shearing force you can apply to the horn of some domesticated animal by means of **Leavitt's V Shape Blade Dehorning Clipper**.

2.5   *Show that* if the coefficient of friction between the block and the plane is 0.25, the force required to just start the block moving **up** the 40° incline is $F = 1.38\ W$ while the force required to hold the block from sliding **down** the plane is $F = 0.487\ W$.
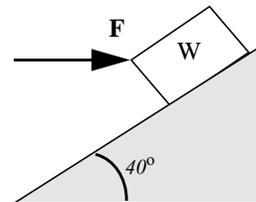
2.6 A block slides down an plane inclined at 20°. *What if* the coefficient of friction is doubled; at what angle of incline will the block begin to slide down the plane?

2.7 *Estimate* the difference in water pressure between the fourth and first floors of bldg. 1.

2.8 *Show that* if, in exercise 2.8, you take a position vector from A to D in computing the moment due to the tension in the cable BD, rather than from A to B as was done in the text, you obtain the same result for the moment of the tension with respect to the point A.

2.9 What do you *need to know* to estimate the torque a driver of a compact automobile must exert to turn the steering wheel (no power steering) while the vehicle is at rest?

2.10 *Estimate* the "live" floor loading in lecture 10-270 during the first 8.01 class of the semester; during the last class of the semester.

2.11 *What if* the urchin of exercise 2.7 pulling the roller grows up and is able to pull on the handle at an angle of θ up from the horizontal; Develop a nondimensional expression for the force she must exert to just start the roller over the curb.

2.12 *Show that*, for the block kept from sliding down the inclined plane by friction alone, that the line of action of the resultant of the normal force (per unit area) distributed over the base of the block must pass through the intersection of the line of action of the weight vector and the line of action of the friction force.What happens when the line of action of the weight falls to the right of point B?
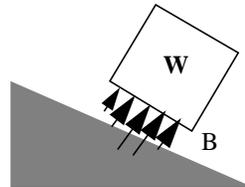
2.13 The rigid beam carries a load P at its right end and is supported at the left end by two (frictionless pins). The pin at the top is pulled upwards and held in place by a cable inclined at a 45 degree angle with the horizontal.
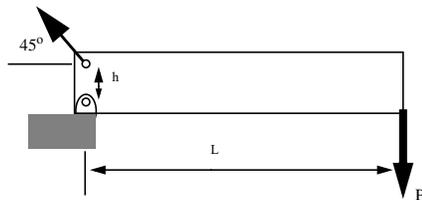
Draw a free body diagram of the beam, isolated from its environment i.e., show all the forces acting on the beam alone; show all relevant dimensions; show a reference cartesian axes system.

2.14   The reaction force at B, of the wall upon the ladder, is greater than, equal to, or less than the weight, W, of the ladder alone?
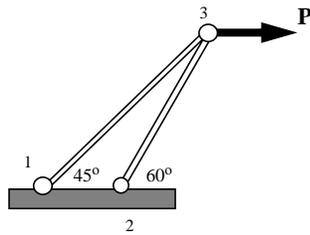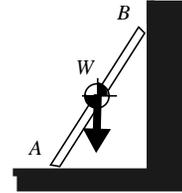
2.15 Isolate pin 3, showing the forces acting on the pin due to the tension (or compression) in the two members.

Find the forces in the members in terms of **P**.

2.15   The truss structure shown carries a load **P** and is supported by a cable, *BC*, and pinned at *D* to the wall. Determine the force in the cable *BC* and the reaction force at *D*.
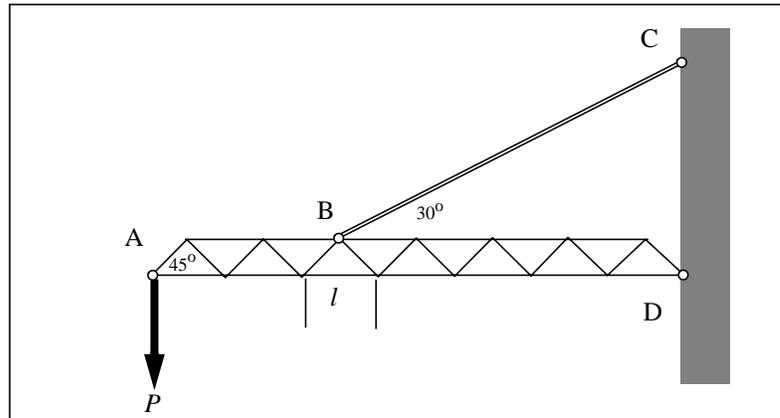
2.16   Isolate pin 4, showing the forces acting on the pin due to the tensions (or compressions) in the three members.

1) Write out consequences of the requirements for static equilibrium of pin 4.

2) Can you solve for the member forces in terms of **P** (and the given angles)?
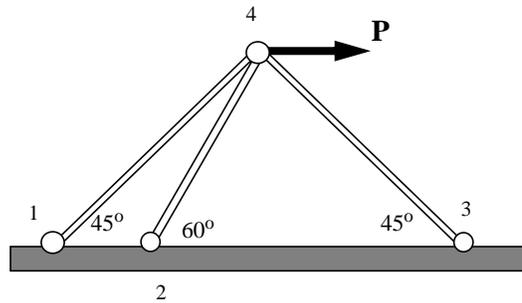
2.17   Estimate the weight of a fully loaded MBTA bus traveling down Mass. Avenue. Estimate an upper bound on the loading per unit length of span of the Harvard Bridge if clogged with buses, both ways.
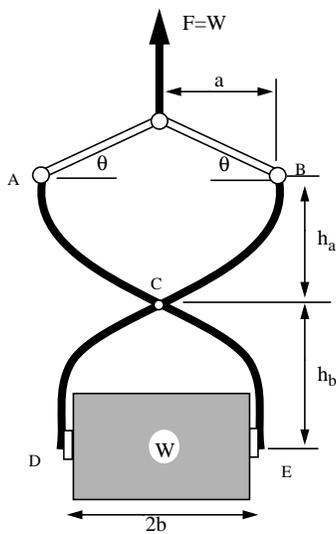
2.18 The vice grip shown is made of two formed steel members pinned at C. Construct an expression for the force component which compresses the block at the bottom in terms of the weight of the block W and the dimensions shown. Express you result in terms of nondimensional factors.

If $\theta = 30°$ $a = b = h_a = h_b$, What must the coefficient of friction be to ensure the block does not slide out of the grip?