Reflections: Design Exercise 4
1.050 Solid Mechanics
Fall 2004

A hollow aluminum shaft, two meters long, must transmit a torque of 20 KNm. and bear a compressive load of 100 KN. The total angle of twist over the full length of the shaft is not to exceed 2.0^o, and of course, we do not want the shaft to yield. Size the outside and inside diameters of the shaft striving to keep the volume of material a minimum.

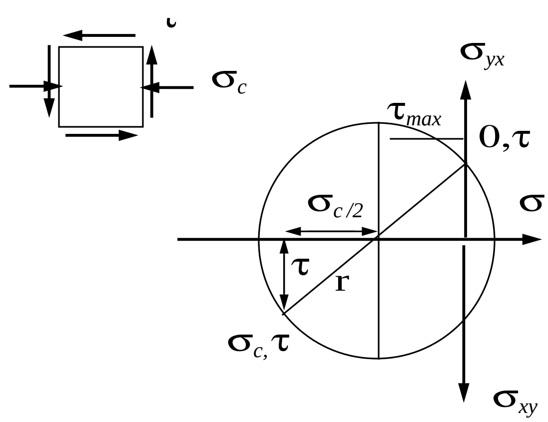
One common deficiency of student work on this exercise was in the way the onset of yield of the aluminum was treated. Most all treated the possibilities of failure due to the compressive load and failure due to twisting of the shaft independently[1]. All wrote expressions for the compressive stress, let it be σ_c, and the shear stress due to torsion, let it be τ, correctly, i.e.,

$$\sigma_c = \frac{F}{A} \qquad and \qquad \tau = \frac{M_t \cdot R_o}{J}$$

where F is *100 KN*, A is the area of the cross-section, $A = \pi(R_o^2 - R_i^2)$, M_t is the applied torque, $M_t = 20$ *KN-m*, and R_o and R_i are the outer and inner radii of the hollow shaft of length $L = 2$ meters. But these two stress components, acting at any point at the outer surface of the shaft, need to be "combined" and the *maximum* shear stress at that "any" point determined in order to apply the maximum shear stress criterion for the onset of yielding. That criterion says that the material will yield at a point when the maximum shear stress at the point equals the maximum shear stress within a bar in tension at the onset of yield, i.e.,

$$\tau|_{max} < \frac{\sigma_{yield}}{2}$$

We find the $\tau|_{max}$ from the transformation relations, or Mohr's circle, for the state of stress at the outer surface of the shaft. We show Mohr's circle at the right: Now the maximum shear stress is just equal to the radius of the circle, which we see is



$$r = \sqrt{\left(\frac{\sigma_c}{2}\right)^2 + (\tau)^2}$$

So our condition on yielding becomes

$$\boxed{\sqrt{\left(\frac{\sigma_c}{2}\right)^2 + (\tau)^2} < \frac{\sigma_{yield}}{2}}$$

1. This may be due to the fact that I spent little time addressing the question of yielding due to combined stress before the exercise was set.

If we treat the compressive stress and the shear stress due to torsion independently we would then require

$$\sigma_C < \sigma_{yield} \qquad and \qquad \tau < \frac{\sigma_{yield}}{2}$$

$$or$$

$$\frac{\sigma_c}{2} < \frac{\sigma_{yield}}{2} \qquad and \qquad \tau < \frac{\sigma_{yield}}{2}$$

$$so$$

$$\left(\frac{\sigma_c}{2}\right)^2 + (\tau)^2 < 2\left(\frac{\sigma_{yield}}{2}\right)^2 \qquad or \qquad \sqrt{\left(\frac{\sigma_C}{2}\right)^2 + (\tau)^2} < \sqrt{2}\cdot\frac{\sigma_{yield}}{2}$$

Thus, if we treat the compressive stress and shear stress independently, we may very well satisfy this last condition yet the shaft may still yield, the case if our max shear is just less than $\sqrt{2}\cdot\frac{\sigma_{yield}}{2}$ but bigger than $\sigma_{yield}/2$.

In your particular situation, for the loadings as specified, whether it does or does not depends upon what you take as a yield stress for Aluminum.

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We seek now this constraint's implications for choice of thickness and (outer) radius. To do this, we make explicit how thickness and radius enter into this relationship. Actually, instead of thickness, t, we will work with the ratio of inner to outer radius, R_i/R_o. They are related by $t = R_o - R_i$.

We have $\sigma_c = \frac{F}{A}$ and $\tau = \frac{M_t \cdot R_o}{J}$ where

$$J = \frac{\pi}{2}\cdot(R_o^{\,4} - R_i^{\,4}) \qquad and \qquad A = \pi(R_o^{\,2} - R_i^{\,2})$$

We rework our constraint introducing non-dimensional expressions for our two design parameters, R_o *and* R_i/R_o. We could just as well have chosen the thickness $t = R_o - R_i$ instead of the ratio of the radii. So we define

$$\boxed{y = \frac{R_o}{L} \qquad \text{and also set} \qquad x = \frac{R_i}{R_o}}$$

and, after considerable careful manipulation of all terms in $\boxed{\sqrt{\left(\frac{\sigma_C}{2}\right)^2 + (\tau)^2} < \frac{\sigma_{yield}}{2}}$ express this as:

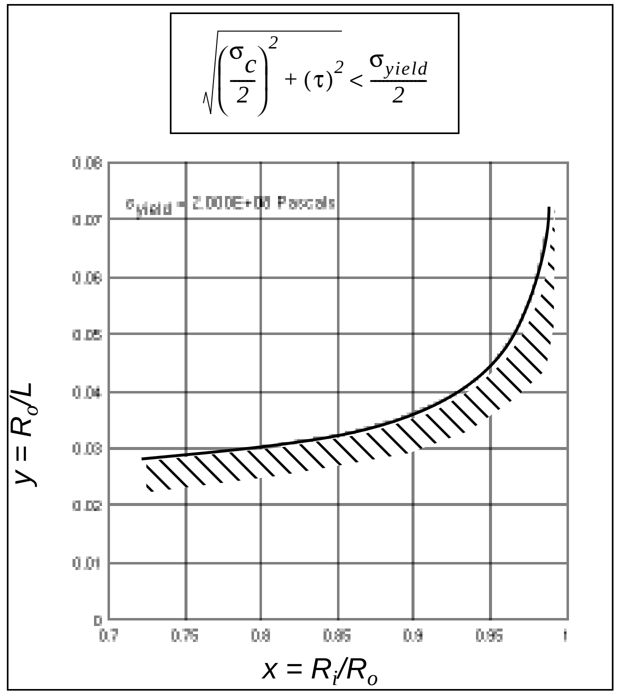
$$\boxed{\left[\frac{C_1^{\,2}}{y^4\cdot(1-x^2)^2} + \frac{C_2^{\,2}}{y^6\cdot(1-x^4)^2}\right]^{1/2} < 1}$$

where the dimensionless constants, for the numbers given, (taking σ_{yield} = *200E06 Pascals*) are

$$\boxed{C_1 = \frac{F}{\pi L^2\sigma_{yield}} = 3.98\times10^{-05} \qquad and \qquad C_1 = \frac{4M_t}{\pi L^3\sigma_{yield}} = 7.96\times10^{-06}}$$

The plot at the right shows the relationship between $y = \frac{R_o}{L}$ and $x = \frac{R_i}{R_o}$ for this constraint to be satisfied. For a given x, the outer radius must lie above the shaded region for the inequality to be satisfied[1].

As the thickness decreases, (and *x* increases), we need to increase the outer radius relative to the length of the shaft, increasing *J* and *A* to keep the maximum shear stress due to the combined torsion and compression, within bounds.



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It remains to consider the other constraint, i.e., the angle of twist must not exceed *2°. Φ< 0.035 radians.*

We have

$$\Phi = \frac{M_t \cdot L}{GJ} < 0.035$$

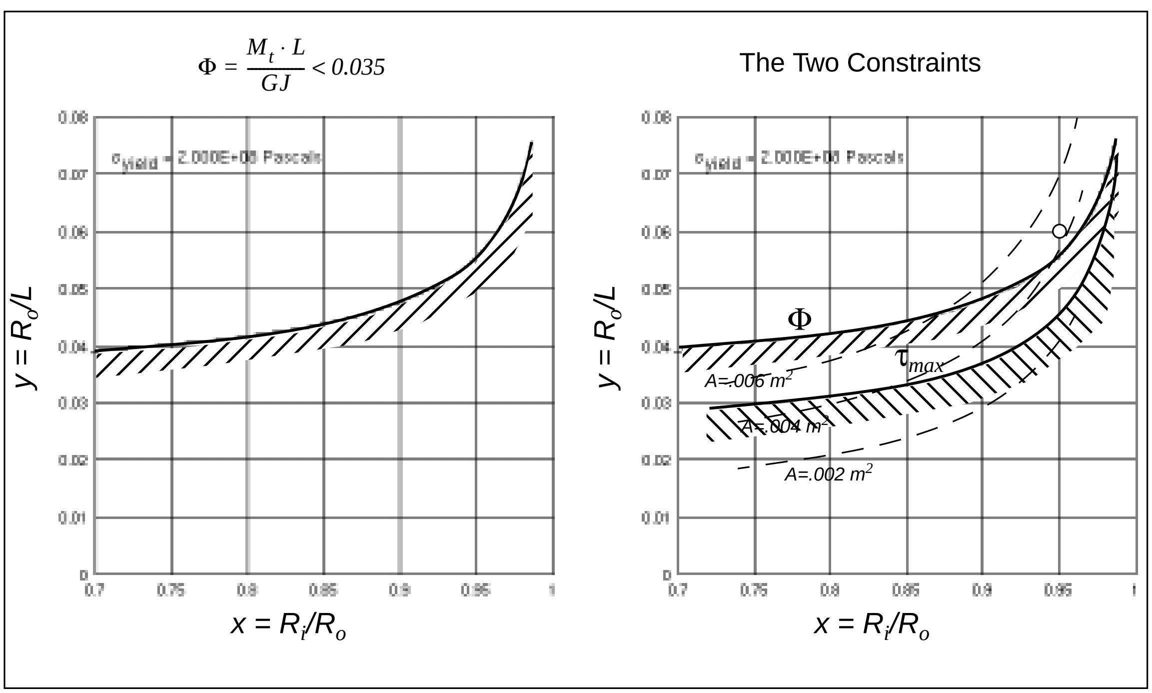
Putting *J* in terms of the radii and manipulating, this condition may be written:

$$R_o^{\ 4} - R_i^{\ 4} > \frac{M_t \cdot L}{G\Phi} \cdot \frac{2}{\pi}$$

which, in terms of non-dimensional expressions for the outer radius and the ratio of inner radius to outer radius, becomes

$$\boxed{y^4 \cdot (1 - x^4) > \frac{M_t}{G\Phi \cdot L^3} \cdot \frac{2}{\pi} = 1.75 \times 10^{-06}}$$

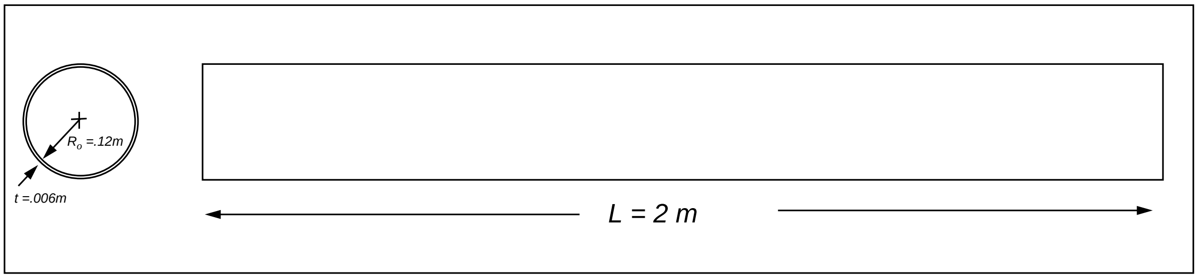
We plot at the left below, the locus of *x,y* points if equality holds. All points above the shaded region satisfy the inequality. In the plot at the right, we show the two curves together.



1. Note: This is not the first plot I constructed. I originally scaled the *y* axis from *0 to.5* but this was not a good presentation of the relationship.

We include also three constant-area contour lines. To minimize area, one would like to reduce the outer radius while, at the same time, reduce the thickness ($x = R_i/R_o$ closer to *1.0*).

We see that the constraint on the angle of rotation dominates, at least for thickness that are not too small (x large). Minimizing area in this case would mean moving up the constraint curve, $\Phi = 2^o$, as far as one can go. Ultimately, we will reach some point where the tube's wall is so thin that buckling will occur.



A possible design choice is the point (circled) at $x = .95$, $y = 0.06$. This gives an outer radius of $R_o = 0.12\ m$ and a wall thickness of $t = .006\ m$.

My check for possible buckling says that we are in trouble if the shear stress due to torsion alone approaches *300E06 Pascals*. But experimental results show that buckling of this system is sensitive to the slightest imperfections. Values obtained from experiment cluster around 75% of this value and even drop below. Taking 50% of this as a "safe value", we should be ok: The shear stress due the torque M_t = *20E03 Nm* is about *40E03 Pascals < 150E03 Pascals*

Shell buckling due to the compressive load is also possible but we are well within the limit due to this constraint.

Some observations:

- Other constraints will no doubt play a role in setting dimensions of the shaft. Available shafts may come in discrete forms. Of course a custom design could always be specified but that is generally much more costly than going with what's available "off the shelf".
- In a case such as this where there are two design parameters to be chosen, a plot showing the boundaries on choices for the two gives a good picture of what is possible. In some cases the constraints will define a closed region.
- The plot of the max. shear stress locus of *x,y* pairs required me to use an iterative method to find each point. This was not difficult, once I had developed a firm grip of the problem, but getting there took a good bit of time ~ 24 hours!
- For the yield stress I chose, the constraint on the angle of twist dominated. This means that even if you did not figure the max. shear stress due to the *combined stress state* (first page here) you would probably be ok. Still, note that assuming the two stresses act independently may lead to trouble.