1.050 Engineering Mechanics

Lecture 4: Stresses and Strength
Stresses and Equilibrium
Discrete Model

1.050 – Content overview

I. Dimensional analysis
   1.   On monsters, mice and mushrooms
   2.   Similarity relations: Important engineering tools

Lectures 1-3
Sept.

II. Stresses and strength
   2.   Stresses and equilibrium
   3.   Strength models (how to design structures, foundations.. against mechanical failure)

Lectures 4-15
Sept./Oct.

III. Deformation and strain
   4.   How strain gages work?
   5.   How to measure deformation in a 3D structure/material?

Lectures 16-19
Oct.

IV. Elasticity
   5.   Elasticity model – link stresses and deformation
   6.   Variational methods in elasticity

Lectures 20-31
Nov.

V. How things fail – and how to avoid it
   7.   Elastic instabilities
   8.   Plasticity (permanent deformation)
   9.   Fracture mechanics

Lectures 32-37
Dec.

1.050 – Content overview

I. Dimensional analysis

II. Stresses and strength

III. Deformation and strain

IV. Elasticity

V. How things fail – and how to avoid it

Content lecture 4

1. **Review: Newton's Laws of Motion**

2. **Application: Discrete Model**
    - Linear Momentum & Dynamic Resultant Theorem
    - Angular Momentum & Dynamic Moment Theorem

3. **Exercise: The Fall of the WTC Towers**
    1. Free Fall Assumption
    2. Discrete Model
    3. From Discrete to Continuum

Goal: Put Newton's Laws to work.

9-11-2001: The Fall of the Towers

North Tower: 8:46 am above 96th floor, failed at 10:28 am
South Tower: 9:03 am above 80th floor, failed at   9:59 am

Immediate question: How did the towers fail?

Three sequential photographs of tower collapse removed due to copyright restrictions.

Physics Background

The Three Laws of Motion of Isaac Newton (1642 – 1727):

1. Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.
2. The change of motion is proportional to the motive force impresses, and is made in the direction of the right line in which that force is impressed.
3. To every action there is always opposed an equal reaction: or, the mutual action of two bodies upon each other are always equal, and directed to contrary parts.

Our Aim:

Translate these Laws into powerful tools of Engineering Mechanics

Dynamic Resultant Theorem: Discrete Mass System

- Linear motion of a mass is quantified by the linear momentum vector:

$$\vec{\wp} = m\vec{V} = m\left(V_1 \vec{e}_1 + V_2 \vec{e}_2 + V_3 \vec{e}_3\right)$$

Theorem 2 (Dynamic Resultant Theorem) *The change of the linear momentum is equal to the sum of external forces:*

$$\left| \frac{d\vec{\wp}}{dt} = \frac{d}{dt}\left(m\vec{V}\right) \overset{def}{=} \vec{F}^{ext} \right| \qquad (2.2)$$

The external force $\vec{F}^{ext} = \vec{F}_1 + \vec{F}_2 + ...$ *is a vector quantity.*

When the mass remains constant in time, that is, when the system is closed, the dynamic resultant theorem yields the inertia force definition $\vec{F}_{ext} = m\vec{a}$, where $\vec{a} = \frac{d}{dt}\vec{V}$ is the acceleration vector.

Dynamic Moment Theorem: Discrete System

- The angular motion of a mass point i is quantified by the angular momentum vect:

$$\vec{x}_i \times \vec{\wp}_i = \vec{x}_i \times m_i \vec{V}_i$$

Theorem 3 (Dynamic Moment Theorem) *The change of the angular motion of a discrete system of $i = 1, N$ particles is equal to the sum of the moments (or torque) generated by external forces:*
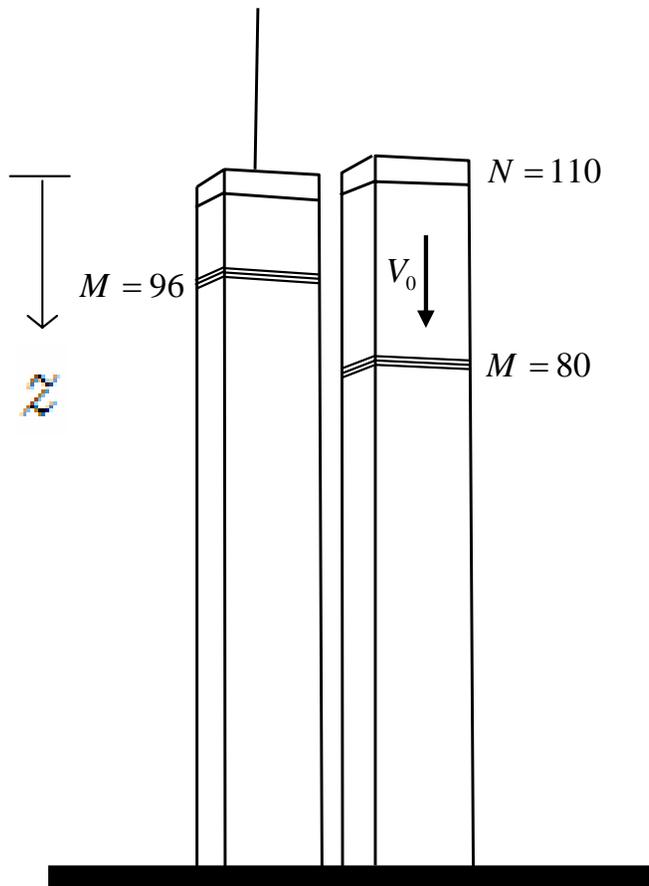
$$\left| \frac{d}{dt} \sum_{i=1}^{N} \left(\vec{x}_i \times m_i \vec{V}_i \right) \stackrel{def}{=} \sum_{i=1}^{N} \vec{x}_i \times \vec{F}_i^{ext} = \sum_{i=1}^{N} \vec{\mathcal{M}}_i^{ext} \right| \qquad (2.4)$$

The external moment $\vec{\mathcal{M}}_i^{ext} = \vec{x}_i \times \vec{F}_i^{ext}$ *is a vector quantity.*

cross product

9-11: engineering questions

- **Free Fall?**

$N = 110$

$M = 96$

V_0

$M = 80$

z

$z - z_0 = Mh$

$h = 3.75\,\text{m}$

- **Dynamic Resultant Theorem:**

$$\frac{d\vec{p}}{dt} = m_0 a\,\vec{e}_z \overset{def}{=} m_0 g\,\vec{e}_z$$

- **Integrate twice + Initial velocity**

$$V_{\text{max}} = V_0\sqrt{1 + M}$$
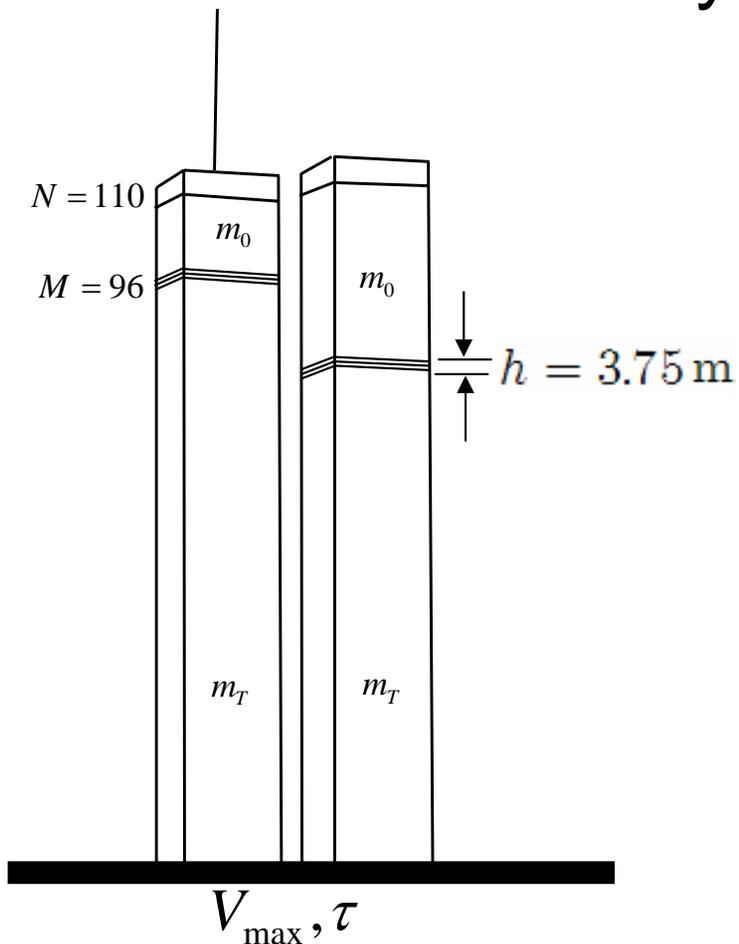
$$\tau = \sqrt{\frac{2h}{g}}\left(\sqrt{1 + M} - 1\right)$$

North Tower: $\tau\,(M = 96) \approx 7.7\,\text{s}$
South Tower: $\tau\,(M = 80) \approx 7.0\,\text{s}$

9-11: engineering questions (cont'd)

- Return to Dimensional Analysis

$N = 110$

m_0

$M = 96$

m_0

$h = 3.75\,\mathrm{m}$

m_T

m_T

$V_{\max}, \tau$

- Problem Formulation

$$\tau = f(g, m_0, m_T, h)$$

- Exponent Matrix (k=3)

	$\lfloor \tau \rfloor$	$\lfloor g \rfloor$	$\lfloor m_0 \rfloor$	$\lfloor m_T \rfloor$	$\lfloor h \rfloor$
L	0	1	0	0	1
M	0	0	1	1	0
T	1	-2	0	0	0

- Pi-Theorem

$$\Pi_0 = \tau \sqrt{\frac{g}{h}} = \mathcal{F}\left(\Pi_1 = \frac{m_0}{m_T} = \frac{N - M}{M}\right)$$

# 9-11: engineering questions (cont'd)

- Kausel's Discrete Mass Formulation

m_0, V_0

$m_1 = m_0 + m$

$V_1 = ?$

$m_2 = m_0 + 2m$

$V_2 = ?$

$m_i = m_0 + im$

$V_i = ?$

$m_{N-M} = m_0 + (N-M)m$

$V_{max} = V_{N-M} = ?$

h
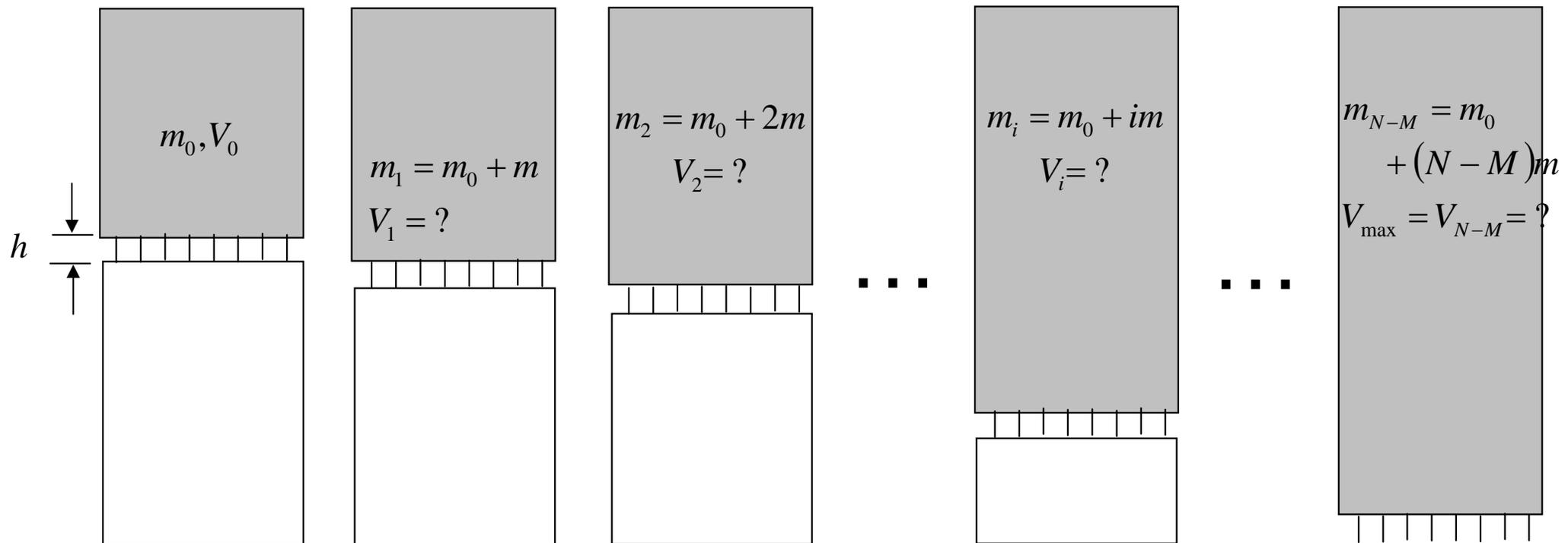
Sequence of 1-story free-falls: when mass collides with floor below, they continue together the free fall until next floor level. There is no resistance to this fall (neither Strength, drag force, etc…)

Application of Dynamic Resultant Theorem

Before Collision
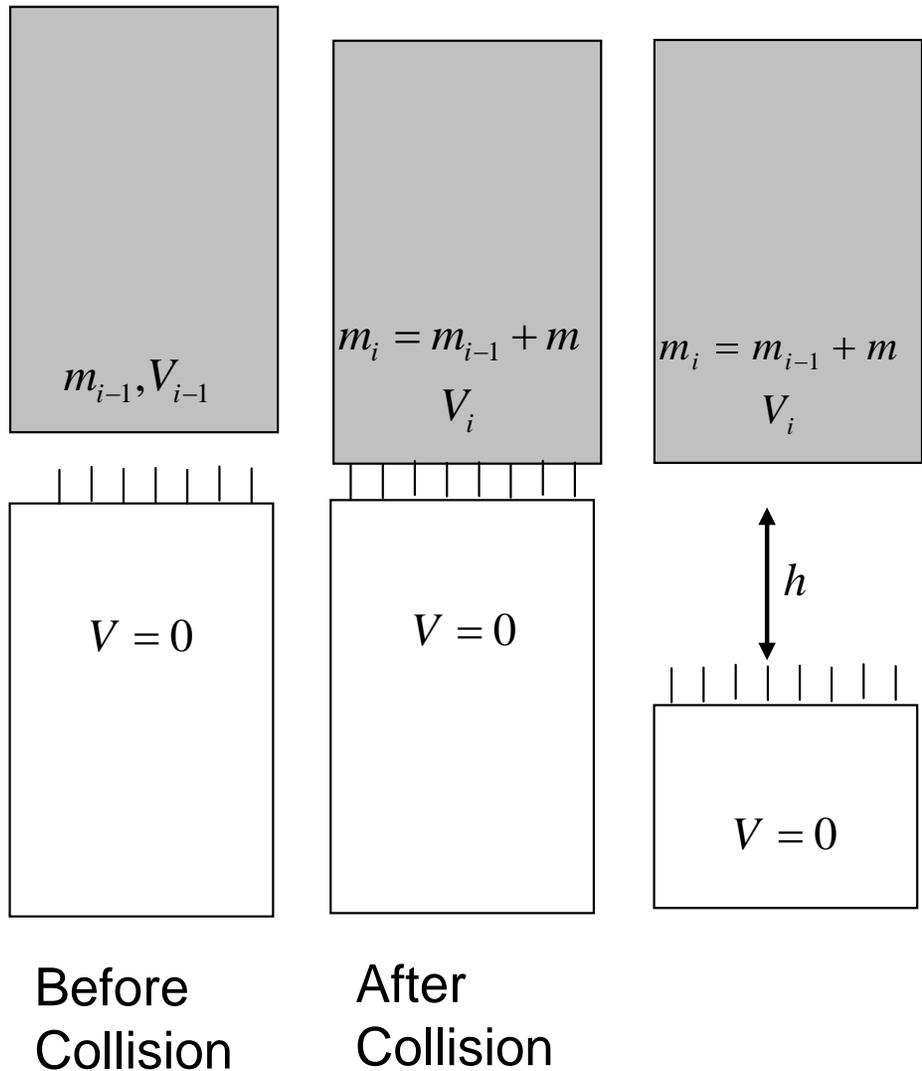
After Collision

- Linear Momentum before collision

$$\vec{p}_{i-1} = m_{i-1} V_{i-1}\, \vec{e}_z$$

- Linear Momentum after collision

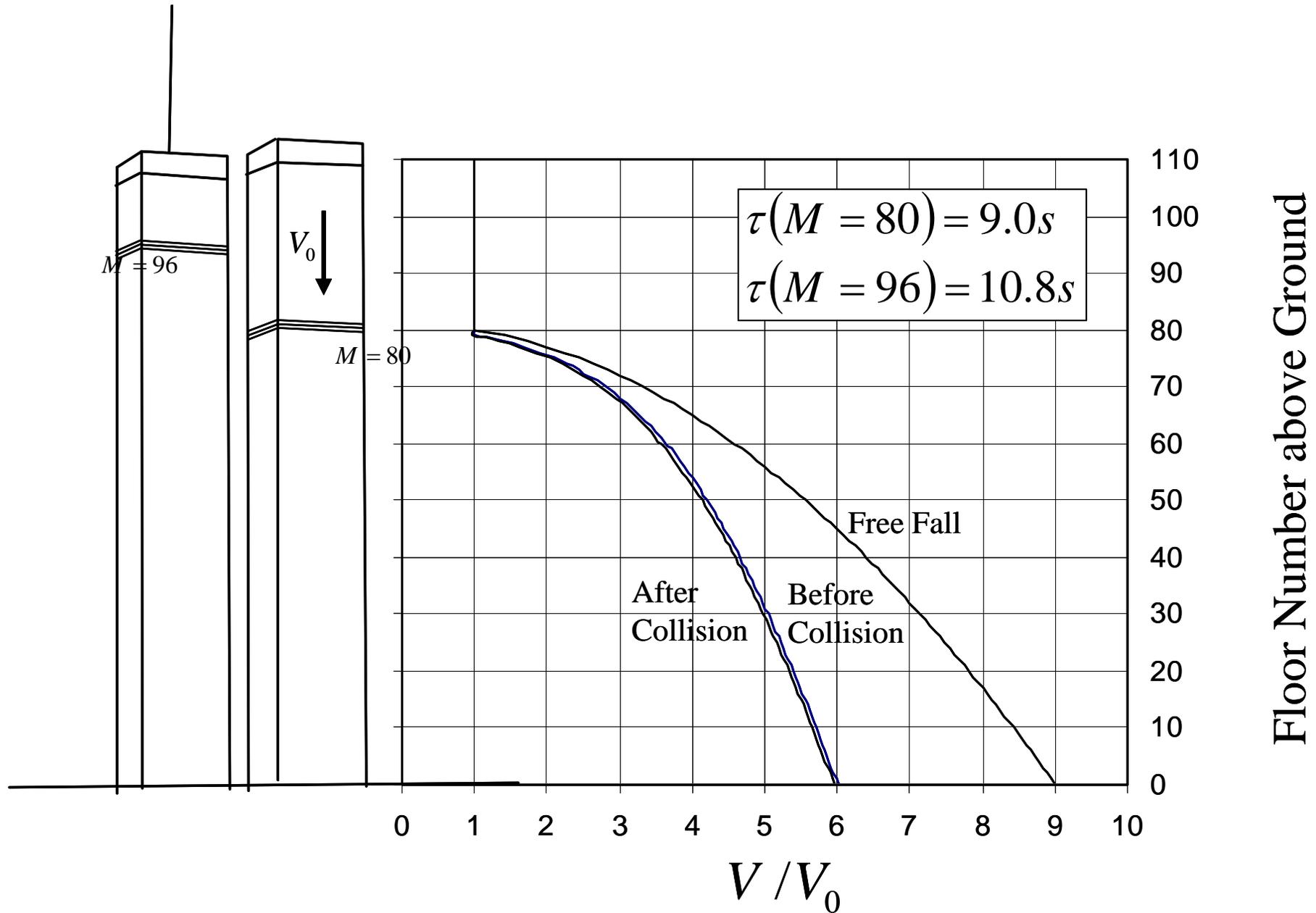$$\vec{p}_{i-1} = m_i V_i\, \vec{e}_z$$

- Instantaneous Conservation of Linear Momentum

$$\delta \vec{p}_i = 0 \Rightarrow V_i = \frac{m_{i-1}}{m_i} V_{i-1}$$

- Time of free fall over inter-story height

$$\Delta t_i = \frac{V_i - V_{i-1}}{g}$$

Results of the Discrete Model

$$\tau(M = 80) = 9.0s$$

$$\tau(M = 96) = 10.8s$$

V_0

$M = 96$

$M = 80$

Free Fall

After Collision

Before Collision

Floor Number above Ground

V / V_0

From the Discrete Model to the Continuum Model

- Discrete Model
  - Discrete mass system

$$m_i = m_0 + i\,m$$

$$\xrightarrow{\; h/H = 1/110 \ll 1 \;}$$

  - Momentum balance at each floor level

$$\delta \vec{p_i} = 0 \Rightarrow V_i = \frac{m_{i-1}}{m_i} V_{i-1}$$

- Continuum Model
  - Continuous mass

$$m(z) = \frac{m}{h}\, z(t)$$

  - Momentum balance on the moving front:

$$\frac{d\vec{p}}{dt} = \frac{d\big(m(z(t))V\big)}{dt}\, \vec{e_z}$$

$$= \frac{m}{h}\left(\dot{z}^2 + z\ddot{z}\right)\vec{e_z} = \frac{m}{h}\, z g\, \vec{e_z}$$

Continuum Approach

- Differential Equation

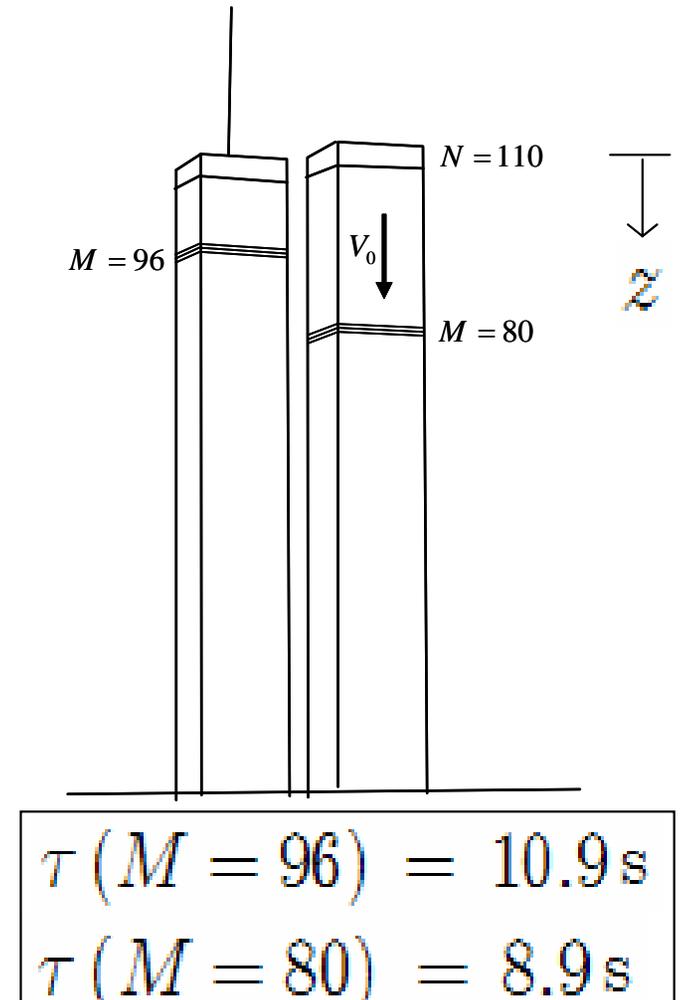$$\dot{z}^2 + z\ddot{z} = zg$$

- Boundary Conditions

$$z(t = 0) = z_0 = (N - M)h$$
$$\dot{z}(t = 0) = V_0 = \sqrt{2gh}$$

- Solution* yields $z(t)$
  - Evaluate for $z(t = \tau) = Nh$

$N = 110$

$M = 96$ V_0

$M = 80$

z

$$\tau(M = 96) = 10.9\,\mathrm{s}$$
$$\tau(M = 80) = 8.9\,\mathrm{s}$$

(*) with MATLAB

9-11: engineering questions (last)

Why did the towers not tilt?

Think: Dynamic Moment Theorem

$$\frac{d}{dt} \sum_{i=1}^{N} \left(\vec{x}_i \times m_i \vec{V}_i \right) \stackrel{def}{=} \sum_{i=1}^{N} \vec{x}_i \times \vec{F}_i^{ext}$$

Photograph of airplane about to strike the south tower removed due to copyright restrictions.

And ask yourself, whether the resulting moment would have been large enough to reach the strength limit of a building designed to withstand the moment generated by forces of a hurricane (weight equivalence of 1000 elephants)

World Trade Center Towers (1973 – 2001)
Engineer: Leslie E. Robertson

Boeing 767 aircraft approaching the South Tower (www)
Max Fuel: 90 m^3 - Total max weight ~ 500 tons
Approaching Speed V ~ 691 km/h (NT) / 810 km/h (ST)