1.017/1.010 Class 18
Small Sample Statistics

Small Samples

When sample size N is small the estimator of a distributional property a (mean, variance, 90 percentile, etc.) is generally not normal.

In this case, the CDF's of the estimate $\hat{a}$ and standardized statistic z (used to derive confidence intervals and hypothesis tests) can be approximated with **stochastic simulation**.

In order to generate random replicates in the stochastic simulation we need to specify the property a (or parameters that are related to it):

For estimating confidence intervals we assume $a = \hat{a}$ (the estimate computed from the actual data).

For testing hypotheses we assume $a = a_0$ (the hypothesized parameter value).

The stochastic simulation uses many N_{rep} random sample replicates, each of length N, to generate N_{rep} estimates. The desired estimate and standardized statistic CDFs are derived from this ensemble of estimates.

<u>Example – Small-sample two-sided confidence Intervals for the mean of an exponential distribution</u>

Consider a small sample that is thought to be drawn from an exponential distribution with unknown parameter a:

$$[x_1, x_2, x_3, x_4, x_5] = [0.05 \quad 1.46 \quad 0.50 \quad 0.72 \quad 0.11\ ]$$

The sample mean is an unbiased estimator of a:

$$\hat{a} = m_x = 0.57$$

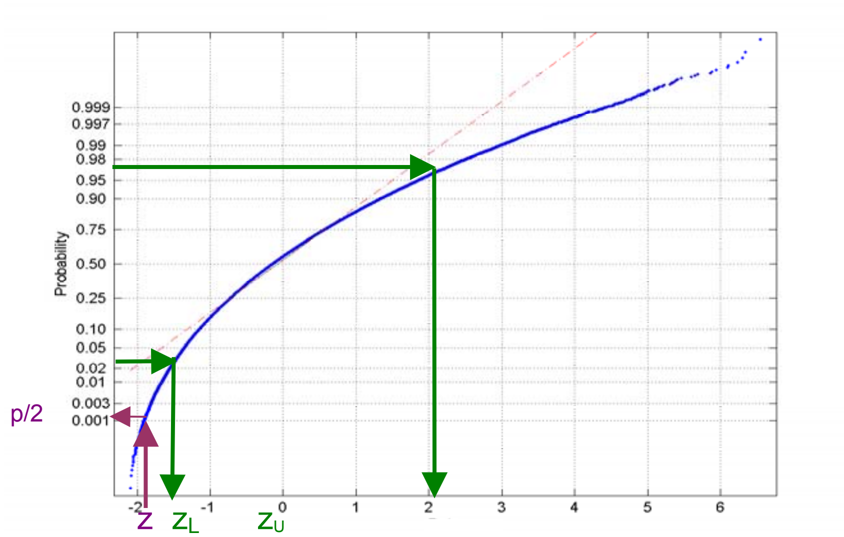
As in the large sample case we derive the a confidence interval from a standardized statistic z. Replicate i of z is:.

$$z^i(\hat{a}^i, m_{\hat{a}}) = \frac{\hat{a}^i - m_{\hat{a}}}{s_{\hat{a}}}$$

where $m_{\hat{a}}$ and $s_{\hat{a}}$ .are the sample mean and standard deviation computed over the ensemble of all estimate replicates $\hat{a}^i$ (e.g. $i = 1,\ldots, Nrep$). Each $\hat{a}^i$ is derived from $\mathrm{N} = 5$ data values obtained from the MATLAB function `exprnd`, with $a = m_x = 0.57$.

The CDF $F_z(z)$ is obtained by plotting the z^i replicates with `cdfplot` or `normplot`.

$F_z(z)$ for this example clearly deviates from the unit normal at both high and low values:



$F_z(z)$ is used as in the large sample case to identify the z_L and z_U values:

$$z_L = F_Z^{-1}\left[\frac{\alpha}{2}\right] \qquad z_U = F_Z^{-1}\left[1-\frac{\alpha}{2}\right]$$

The small-sample double-sided 95% confidence interval for a is approximately:

$$a_L = \hat{a} - z_U s_{\hat{a}} = 0.56 \text{ - } (+2.1)(0.255) = 0.02$$
$$a_U = \hat{a} - z_L s_{\hat{a}} = 0.56 \text{ - } (-1.5)(0.255) = 0.94$$
$$0.02 \le a \le 0.94$$

For comparison, the 95% doubled-sided large-sample (normal) interval:

$$0.06 \le a \le +1.06$$

The difference is slight considering the small sample size. The difference in the small and large-sample 99% confidence intervals is greater.

<u>Example – Small-sample two-sided test of a hypothesis about the mean of an exponential distribution</u>

Consider in the above example the hypothesis:

$$H0: a=1.0$$

We can derive the rejection region and p value for this hypothesis with a stochastic simulation similar to the performed above except that we use $a = a_0 = 1.0$ in `exprnd` and derive $F_z(z)$ from replicates defined as follows:

$$z^i(\hat{a}^i, a_0) = \frac{\hat{a}^i - a_0}{s_{\hat{a}}} = \frac{\hat{a}^i - 1.0}{s_{\hat{a}}}$$

In this case the $F_z(z)$ plot is the same as the one shown above.

The test statistic obtained from the observed sample mean is:

$$z(\hat{a}, a_0) = \frac{\hat{a} - a_0}{s_{\hat{a}}} = \frac{0.56 - 1.0}{0.255} = -1.73$$

This gives a p value of approximately 0.004 (see figure), leading us reject the hypothesis.

Special Case: Normally Distributed Samples

If random sample(s) are **normally distributed** it is possible to derive the exact **small sample** CDFs of certain **standardized statistics**

Two-sided Confidence Intervals for Small Normally Distributed Samples

Confidence Intervals for the Mean $E(\mathbf{x})$:

Standardized statistic: $$\boldsymbol{t}(m_x, a) = \frac{\boldsymbol{m}_x - a}{\frac{s_x}{\sqrt{N}}}$$

This has a ***t* distribution** with $n=N-1$ degrees of freedom.

Confidence interval:

$$m_x - F_{t,v}^{-1}\left(1-\frac{\alpha}{2}\right)\frac{s_x}{\sqrt{N}} \le a \le m_x - F_{t,v}^{-1}\left(\frac{\alpha}{2}\right)\frac{s_x}{\sqrt{N}}$$

Evaluate $F_{t,v}^{-1}$ with MATLAB function `tinv`.

Confidence Intervals for the Variance *Var*(*x*):

Standardized statistic: $\chi^2(\boldsymbol{s}_x^2, \sigma_x^2) = \dfrac{(N-1)\boldsymbol{s}_x^2}{\sigma_x^2}$

This has a **Chi-squared distribution** with $v = N - 1$ degrees of freedom.

Confidence interval:

$$\frac{(N-1)s_x^2}{F_{\chi 2,v}^{-1}\left(1-\frac{\alpha}{2}\right)} \le Var[\boldsymbol{x}] \le \frac{(N-1)s_x^2}{F_{\chi 2,v}^{-1}\left(\frac{\alpha}{2}\right)}$$

Evaluate $F_{\chi 2,v}^{-1}$ with MATLAB function `chi2inv`.

Two-sided Hypothesis Tests for Small Normally Distributed Samples

Hypothesis Tests about the Mean *E*(x):

$H0 : E(x) = a_0$

Use t test statistic $(v = N - 1)$: $t(m_x, a_0) = \dfrac{m_x - a_0}{\frac{s_x}{\sqrt{N}}}$

p value:

$$\frac{p}{2} = F_{t,v}[t(m_x, a_0)] \text{ for } F_{t,v} \le 0.5$$

$$1 - \frac{p}{2} = F_{t,v}[t(m_x, a_0)] \text{ for } F_{t,v} > 0.5$$

Evaluate $F_{t,v}$ with MATLAB function `tcdf`.

Hypothesis Tests about the Variance *Var*(*x*)

$$H0: Var(x) = \sigma_x^2 = a_0$$

Use Chi-squared test statistic $(\nu = N-1)$: $\chi^2(s_x^2, a_0) = \dfrac{(N-1)s_x^2}{a_0}$

$$\frac{p}{2} = F_{\chi 2,\nu}[\chi^2(s_x^2, a_0)] \text{ for } F_{\chi 2,\nu} \le 0.5$$

$$1 - \frac{p}{2} = F_{\chi 2,\nu}[\chi^2(s_x^2, a_0)] \text{ for } F_{\chi 2,\nu} > 0.5$$

Evaluate $F_{t,\nu}$ with MATLAB function `chi2cdf`.