Slowly Applied (Adiabatic) Perturbation

All of our perturbations so far have been applied suddenly at $t > t_0$ (step function)

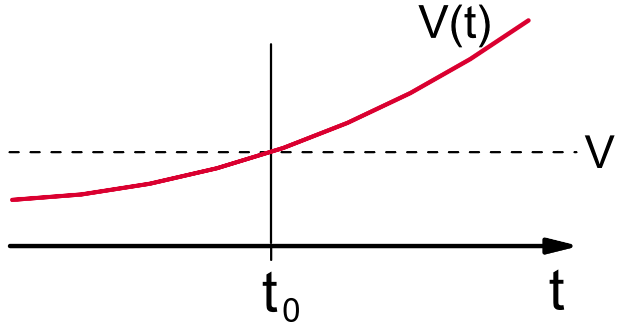
$$V(t) = \theta(t - t_0)V(t)$$

This leads to unphysical consequences—you generally can't turn on a perturbation fast enough to appear instantaneous. Since first-order P.T. says that the transition amplitude is related to the Fourier Transform of the perturbation, this leads to additional Fourier components in the spectral dependence of the perturbation—even for a monochromatic perturbation!

So, let's apply a perturbation slowly . . .

$$V(t) = V\, e^{\eta t}$$

η: small and positive



The system is prepared in state $|\ell\rangle$ at $t = -\infty$. Find $P_k(t)$.

$$b_k = \langle k|U_I|\ell\rangle = \frac{-i}{\hbar}\int_{-\infty}^{t} d\tau\, e^{i\omega_{k\ell}\tau}\langle k|V|\ell\rangle e^{\eta\tau}$$

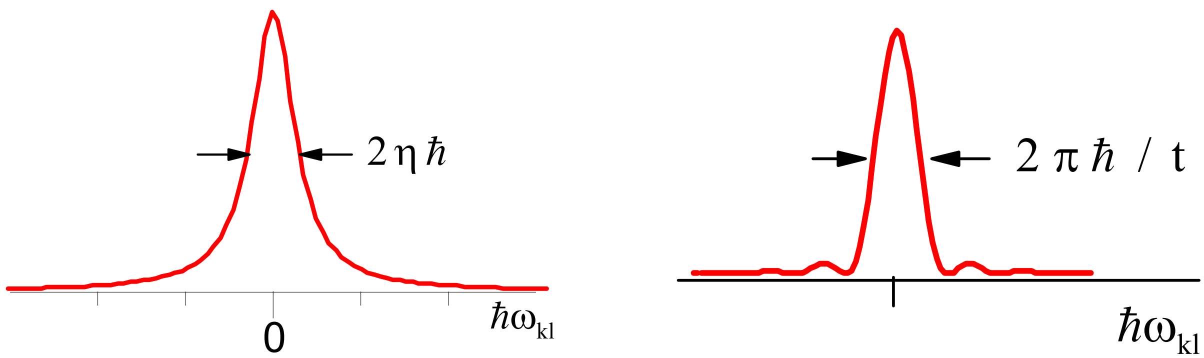
$$b_k = \frac{-iV_{k\ell}}{\hbar}\frac{\exp[\eta t + i\omega_{k\ell}t]}{\eta + i\omega_{k\ell}}$$

$$= V_{k\ell}\frac{\exp[\eta t + i(E_k - E_\ell)t/\hbar]}{E_k - E_\ell + i\eta\hbar}$$

$$P_k = |b_k|^2 = \frac{|V_{k\ell}|^2}{\hbar^2}\frac{\exp[2\eta t]}{\eta^2 + \omega_{k\ell}^2} = \frac{|V_{k\ell}|^2\exp[2\eta t]}{(E_k - E_\ell)^2 + (\eta\hbar)^2}$$

This is a Lorentzian lineshape in $\omega_{k\ell}$ with width $2\eta\hbar$.

Gradually Applied Perturbation **Step Response Perturbation**



The gradually turned on perturbation has a width dependent on the turn-on rate, and is independent of time. (The amplitude grows exponentially in time.) Notice, there are no nodes in P_k.

η^{-1} is the effective turn-on time of the perturbation:

Now, let's calculate the transition rate:

$$w_{kl} = \frac{\partial P_k}{\partial t} = \frac{|V_{k\ell}|^2}{\hbar^2} \frac{2\eta e^{2\eta t}}{\eta^2 + \omega_{k\ell}^2}$$

Look at the adiabatic limit; $\eta \to 0$.

$$\text{setting } e^{2\eta t} \to 1; \text{ and using } \lim_{\eta \to 0} \frac{\eta}{\eta^2 + \omega_{k\ell}^2} = \pi\delta(\omega_{k\ell})$$

$$w_{k\ell} = \frac{2\pi}{\hbar^2} |V_{k\ell}|^2 \delta(\omega_{k\ell}) = \frac{2\pi}{\hbar} |V_{k\ell}|^2 \delta(E_k - E_\ell)$$

We get Fermi's Golden Rule—independent of how perturbation is introduced!

<u>If we gradually apply the Harmonic Perturbation,</u>

$$V(t) = V\, e^{\eta t} \cos\omega t$$

$$b_k = \frac{-i}{\hbar}\int_{-\infty}^{t} d\tau\, V_{k\ell}\, e^{i\omega_{k\ell}\tau + \eta\tau}\left[\frac{e^{i\omega\tau} + e^{-i\omega\tau}}{2}\right]$$

$$= \frac{V_{k\ell}}{2\hbar} e^{\eta t}\left[\frac{e^{i(\omega_{k\ell}+\omega)t}}{-(\omega_{k\ell}+\omega)+i\eta} + \frac{e^{i(\omega_{k\ell}-\omega)t}}{-(\omega_{k\ell}-\omega)+i\eta}\right]$$

Again, we have a resonant and anti-resonant term, which are now broadened by η.

If we only consider <u>absorption</u>:

$$P_k = \left|b_k\right|^2 = \frac{\left|V_{k\ell}\right|^2}{4\hbar^2}\, e^{2\eta t}\, \frac{1}{(\omega_{k\ell}-\omega)^2 + \eta^2}$$

which is the Lorentzian lineshape centered at $\omega_{k\ell} = \omega$ with width $\Delta\omega = 2\eta$.

Again, we can calculate the adiabatic limit, setting $\eta \to 0$. We will calculate the rate of transitions $\omega_{k\ell} = \partial P_k / \partial t$. <u>But</u> let's restrict ourselves to long enough times that the harmonic perturbation has cycled a few times (this allows us to neglect cross terms) $\rightarrow$ resonances sharpen.

$$\mathrm{w}_{k\ell} = \frac{\pi}{2\hbar^2}\left|V_{k\ell}\right|^2\left[\delta(\omega_{k\ell}-\omega) + \delta(\omega_{k\ell}+\omega)\right]$$