5.73 Solution 1, Fall 05
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1. a. $|L\rangle = \frac{1}{\sqrt{2}}(|\vec{x}\rangle - i|\vec{y}\rangle)$ $\quad |R\rangle = \frac{1}{\sqrt{2}}(|\vec{x}\rangle + i|\vec{y}\rangle)$

This set is orthonormal if it is normalized:

$$\langle L|L\rangle = \tfrac{1}{2}(\langle\vec{x}| + i\langle\vec{y}|)(|\vec{x}\rangle - i|\vec{y}\rangle) = \tfrac{1}{2}(\cancel{\langle\vec{x}|\vec{x}\rangle}^{1} + i\cancel{\langle\vec{y}|\vec{x}\rangle}^{0} - i\cancel{\langle\vec{x}|\vec{y}\rangle}^{0} + \cancel{\langle\vec{y}|\vec{y}\rangle}^{1}) = 1$$

$$\langle R|R\rangle = \tfrac{1}{2}(\langle\vec{x}| - i\langle\vec{y}|)(|\vec{x}\rangle + i|\vec{y}\rangle) = \tfrac{1}{2}(\cancel{\langle\vec{x}|\vec{x}\rangle}^{1} - i\cancel{\langle\vec{y}|\vec{x}\rangle}^{0} + i\cancel{\langle\vec{x}|\vec{y}\rangle}^{0} + \cancel{\langle\vec{y}|\vec{y}\rangle}^{1}) = 1$$

and orthogonal.

$$\langle L|R\rangle = \tfrac{1}{2}(\langle\vec{x}| + i\langle\vec{y}|)(|\vec{x}\rangle + i|\vec{y}\rangle) = \tfrac{1}{2}(\cancel{\langle\vec{x}|\vec{x}\rangle}^{1} + i\cancel{\langle\vec{y}|\vec{x}\rangle}^{0} + i\cancel{\langle\vec{x}|\vec{y}\rangle}^{0} - \cancel{\langle\vec{y}|\vec{y}\rangle}^{1}) = 0$$

$$\langle R|L\rangle = \langle L|R\rangle^{*} = (0)^{*} = 0$$

b. $|\psi\rangle = a|\vec{x}\rangle + b|\vec{y}\rangle = a'|L\rangle + b'|R\rangle = \frac{a'}{\sqrt{2}}(|\vec{x}\rangle - i|\vec{y}\rangle) + \frac{b'}{\sqrt{2}}(|\vec{x}\rangle + i|\vec{y}\rangle) = \frac{a'+b'}{\sqrt{2}}|\vec{x}\rangle + i\frac{b'-a'}{\sqrt{2}}|\vec{y}\rangle$

$\therefore\ a = \frac{a'+b'}{\sqrt{2}}$ and $b = i\frac{b'-a'}{\sqrt{2}}$

This transformation is possible because both the $\vec{x},\vec{y}$ and the L, R bases span the 2-dimensional space.

c. The left-circular polarization filter, $\hat{F}_L$ applied to an arbitrary state should yield the $|L\rangle$ component: $\hat{F}_L(\alpha|L\rangle + \beta|R\rangle) = \alpha|L\rangle$; $\hat{F}_L = |L\rangle\langle L| = \frac{1}{2}(|\vec{x}\rangle - i|\vec{y}\rangle)(\langle\vec{x}| + i\langle\vec{y}|)$

$$= \tfrac{1}{2}(|\vec{x}\rangle\langle\vec{x}| - i|\vec{y}\rangle\langle\vec{x}| + i|\vec{x}\rangle\langle\vec{y}| + |\vec{y}\rangle\langle\vec{y}|)$$

Similarly: $\hat{F}_R(\alpha|L\rangle + \beta|R\rangle) = \beta|R\rangle$: $\hat{F}_R = |R\rangle\langle R| = \frac{1}{2}(|\vec{x}\rangle + i|\vec{y}\rangle)(\langle\vec{x}| - i\langle\vec{y}|)$

$$= \tfrac{1}{2}(|\vec{x}\rangle\langle\vec{x}| + i|\vec{y}\rangle\langle\vec{x}| - i|\vec{x}\rangle\langle\vec{y}| + |\vec{y}\rangle\langle\vec{y}|)$$

The probability that an $\vec{x}$ polarized photon will pass through a left circular filter is:

$$\langle\vec{x}|\hat{F}_L|\vec{x}\rangle = \tfrac{1}{2}\langle\vec{x}|(|\vec{x}\rangle\langle\vec{x}| - i|\vec{y}\rangle\langle\vec{x}| + i|\vec{x}\rangle\langle\vec{y}| + |\vec{y}\rangle\langle\vec{y}|)|\vec{x}\rangle = \tfrac{1}{2}\langle\vec{x}|(|\vec{x}\rangle - i|\vec{y}\rangle) = \tfrac{1}{2}$$

Thus, there is a 50% chance of an $\vec{x}$ polarized photon passing through the left filter.

d. $\hat{O}_i\hat{O}_j = -\hat{O}_j\hat{O}_i = \frac{i}{2}\hat{O}_k$ for $i,j,k = (1,2,3); (2,3,1); (3,1,2)$ and

$\hat{O}_1 = \frac{1}{2}(|L\rangle\langle R| + |R\rangle\langle L|)$ $\quad \hat{O}_2 = \frac{i}{2}(|L\rangle\langle R| - |R\rangle\langle L|)$ $\quad \hat{O}_3 = \frac{1}{2}(|R\rangle\langle R| - |L\rangle\langle L|)$

I: $\hat{O}_1\hat{O}_2 = (\frac{1}{2})(\frac{i}{2})(0 - |L\rangle\langle L| + |R\rangle\langle R| + 0) = \frac{i}{4}(-|L\rangle\langle L| + |R\rangle\langle R|)$

II: $\hat{O}_1\hat{O}_3 = (\frac{1}{2})(\frac{1}{2})(|L\rangle\langle R| - 0 + 0 - |R\rangle\langle L|) = \frac{1}{4}(|L\rangle\langle R| - |R\rangle\langle L|)$

III: $\hat{O}_2\hat{O}_1 = (\frac{i}{2})(\frac{1}{2})(0 + |L\rangle\langle L| - |R\rangle\langle R| - 0) = \frac{i}{4}(|L\rangle\langle L| - |R\rangle\langle R|)$

IV: $\hat{O}_2\hat{O}_3 = (\frac{i}{2})(\frac{1}{2})(|L\rangle\langle R| + |R\rangle\langle L| + 0 - 0) = \frac{i}{4}(|L\rangle\langle R| + |R\rangle\langle L|)$

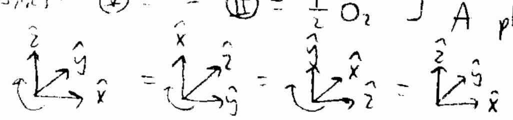
V: $\hat{O}_3\hat{O}_1 = (\frac{1}{2})(\frac{1}{2})(0 - |L\rangle\langle R| + |R\rangle\langle L| + 0) = \frac{1}{4}(|R\rangle\langle L| - |L\rangle\langle R|)$

VI: $\hat{O}_3\hat{O}_2 = (\frac{1}{2})(\frac{i}{2})(0 - |R\rangle\langle L| + 0 - |L\rangle\langle R|) = \frac{i}{4}(-|R\rangle\langle L| - |L\rangle\langle R|)$

$(1,2,3)$: I $= -$ III $= \frac{i}{2}\hat{O}_3$
$(2,3,1)$: IV $= -$ VI $= \frac{i}{2}\hat{O}_1$
$(3,1,2)$: V $= -$ II $= \frac{i}{2}\hat{O}_2$

These combinations are cyclic permutations of one another: 1-2-3 = 2-3-1 = 3-1-2 = 1-2-3

A physical example of another cyclically permuting set is $\hat{x}\ \hat{y}\ \hat{z}$:



No physics should change if space is rotated.

e. $|\psi\rangle\langle\psi| = (a|L\rangle + b|R\rangle)(a^*\langle L| + b^*\langle R|) = |a|^2 |L\rangle\langle L| + |b|^2 |R\rangle\langle R| + ab^*|L\rangle\langle R| + ba^*|R\rangle\langle L|$

$= C_0 |R\rangle\langle R| + C_0 |L\rangle\langle L| + 0 + 0$

$+ 0 + 0 + \frac{C_1}{2}|L\rangle\langle R| + \frac{C_1}{2}|R\rangle\langle L|$

$+ 0 + 0 + \frac{C_2 i}{2}|L\rangle\langle R| - \frac{C_2 i}{2}|R\rangle\langle L|$

$+ \frac{C_3}{2}|R\rangle\langle R| - \frac{C_3}{2}|L\rangle\langle L|$

$$= \underbrace{\left(C_0 + \frac{C_3}{2}\right)}_{|a|^2}|R\rangle\langle R| + \underbrace{\left(C_0 - \frac{C_3}{2}\right)}_{|b|^2}|L\rangle\langle L| + \underbrace{\left(\frac{C_1}{2} + i\frac{C_2}{2}\right)}_{ab^*}|L\rangle\langle R| + \underbrace{\left(\frac{C_1}{2} - i\frac{C_2}{2}\right)}_{ba^*}|R\rangle\langle L|$$

$\frac{|a|^2+|b|^2}{2} = C_0 \qquad |a|^2 - |b|^2 = C_3$

$a^*b + b^*a = C_1 \qquad \frac{ab^* - a^*b}{i} = C_2$

$$\therefore |\psi\rangle\langle\psi| = \left(\frac{|a|^2+|b|^2}{2}\right)\hat{O}_0 + (a^*b + b^*a)\hat{O}_1 + \left(\frac{ab^* - a^*b}{i}\right)\hat{O}_2 + (|a|^2 - |b|^2)\hat{O}_3$$

Yes, any polarization operator can be written in the $\hat{O}_i$ operator basis.

2.a. Since the total momentum is zero, $\vec{k}_1 + \vec{k}_2 = 0$; $\vec{k}_2 = -\vec{k}_1$

b. The first photon can be arbitrarily polarized as long as the polarization of the second photon is opposite

$|\varphi\rangle = \alpha|\vec{k}, L; -\vec{k}, R\rangle + \beta|\vec{k}, R; -\vec{k}, L\rangle$; $|\alpha|^2 + |\beta|^2 = 1$

c. Let $\hat{F}_{L,2}$ only act on photon 2 : $\hat{F}_{L,2}|\varphi\rangle = \beta|\vec{k}, R; -\vec{k}, L\rangle$

$\langle\varphi|\hat{F}_{L,2}|\varphi\rangle = (\beta^*\langle\vec{k}, R; -\vec{k}, L|)(\beta|\vec{k}, R; -\vec{k}, L\rangle) = |\beta|^2$

d. Before the experiment is conducted, we rationalize the probability of finding photon 1 to be left polarized is $|\alpha|^2$.

$\therefore |\beta|^2 = 1 - |\alpha|^2$, which is not necessarily zero!

After the experiment is done, we know $\alpha = 1$ and thus $\beta = 0$, because momentum and angular momentum are always conserved.

Thus, either

(A) we did not completely determine the state of the system beforehand - no probabilities!

or (B) Measuring photon 1 affected photon 2. Since nowhere was the distance between photons specified, this could imply faster than light information transfer.

This is the Einstein, Podolsky, Rosen paradox (EPR) (see Phys. Rev 47, 777-780, 1935)