Problem Set #2 for 5.72 (spring 2012) Cao

I. A diffusive oscillator satisfies the Fokker-Planck equation

$$\frac{\partial p}{\partial t} = D\frac{\partial^2}{\partial x^2}p + \gamma\frac{\partial}{\partial x}(xp),$$

with $\gamma = Dm\omega^2\beta$.

1) Find the equilibrium distribution.
2) Show $\bar{x} = x_0 e^{-\gamma t}$.
3) Assume that $p$ takes the form of

$$p(x_0, x, t) = \frac{1}{\sqrt{2\pi\alpha(t)}} e^{-\frac{(x - x_0 e^{-\gamma t})^2}{2\alpha(t)}}.$$

Find $\alpha(t)$ explicitly.

4) Confirm that solution (2) satisfies equation (1).

II. Consider one-dimensional diffusion under the action of a constant force $F$.

1) Write the Fokker-Planck equation for the diffusive particle and show $\zeta\bar{v} = F$.
2) Show that the solution to the F-P equation is

$$p(x_0, x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left[-\frac{(x - x_0 - D\beta Ft)^2}{4Dt}\right].$$

3) Show that the solution satisfies the stationary condition

$$\int p_{eq}(x_0)p(x_0, x, t)dx_0 = p_{eq}(x),$$

where $p_{eq}(x) = e^{\beta Fx}$ is the equilibrium distribution.

III. *Define the Fourier transform as $\tilde{p}(k,t) = \int e^{-ikx}p(x, x_0, t)dx$.

1) Show that the F-P equation for the diffusive oscillator can be written as

$$\frac{\partial \tilde{p}}{\partial t} = -Dk^2\tilde{p} - \gamma k\frac{\partial}{\partial k}\tilde{p}.$$

2) Solve for $\tilde{p}(k,t)$ with the initial condition $\tilde{p}(k, t = 0) = e^{-ikx_0}$.
3) Give $p(x_0, x, t)$ explicitly.

IV. Diffusive oscillator with a sink at $x_s = 0$.

1) Solve for $p(x_0, x, t)$ subject to the initial condition $p(x_0, x, 0) = \delta(x_0 - x)$ and the absorbing boundary condition $p(x_0, x_s, t) = 0$.
2) Find the mean first passage time from $x_0$ to $x_s$.