Lesson 5: Heated Tank

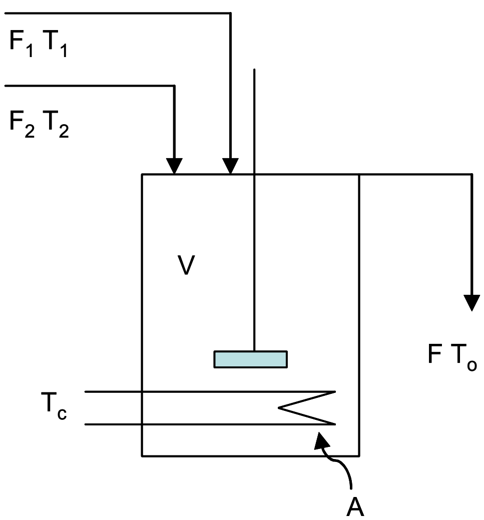
5.0 context and direction

From Lesson 3 to Lesson 4, we increased the dynamic order of the process, introduced the Laplace transform and block diagram tools, took more account of equipment, and discovered how control can produce instability. Now we change the process: our system models have previously depended on material balances, but now we will write the energy balance. We will also introduce the integral mode of control in the algorithm.

DYNAMIC SYSTEM BEHAVIOR

5.1 a heated tank

We consider a tank that blends and heats two inlet streams. The heating medium is a condensing vapor at temperature T_c in a heat exchanger of surface area A.



For the present, we continue to assume constant mass in an overflow tank. Writing the material balance,

$$\rho F_1 + \rho F_2 = \rho F \tag{5.1-1}$$

This is not yet the time for complications: we will approximate the physical properties of the liquid (density, heat capacity, etc.) as constants. We will also simplify the problem by assuming that the flow rates remain constant in time. The energy balance is

$$\frac{d}{dt}\left(\rho V C_p (T_o - T_{ref})\right) = \rho F_1 C_p (T_1 - T_{ref}) + \rho F_2 C_p (T_2 - T_{ref}) + UA(T_c - T_o) - \rho F C_p (T_o - T_{ref}) \tag{5.1-2}$$

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where the overall heat transfer coefficient is U and the *thermodynamic* reference is T_{ref}. We identify a steady-state *operating* reference condition with all variables at their desired values.

$$\frac{d}{dt}\left(\rho V C_p (T_{or} - T_{ref})\right) = 0 = \rho F_1 C_p (T_{1r} - T_{ref}) + \rho F_2 C_p (T_{2r} - T_{ref}) + UA(T_{cr} - T_{or}) - \rho F C_p (T_{or} - T_{ref}) \tag{5.1-3}$$

We subtract (5.1-3) from (5.1-2), define deviation variables, and rearrange to standard form.

$$\frac{\rho V C_p}{\rho F C_p + UA}\frac{dT_o'}{dt} + T_o' = \frac{\rho F_1 C_p}{\rho F C_p + UA}T_1' + \frac{\rho F_2 C_p}{\rho F C_p + UA}T_2' + \frac{UA}{\rho F C_p + UA}T_c' \tag{5.1-4}$$

To make some sense of the equation coefficients, define the tank residence time

$$\tau_R = \frac{V}{F} \tag{5.1-5}$$

and a ratio of the capability for heat transfer to the capability for enthalpy removal by flow.

$$\beta = \frac{UA}{\rho F C_p} \tag{5.1-6}$$

β thus indicates the importance of heat transfer in the mixing of the fluids. We now use (5.1-5) and (5.1-6) to define the dynamic parameters: time constant and gains.

$$\tau = \frac{\tau_R}{1+\beta} \tag{5.1-7}$$

Thus the dynamic response of the tank temperature to disturbances is faster as heat transfer capability (β) becomes more significant. For no heat transfer ($\beta = 0$) the time constant is equal to the residence time.

$$K_1 = \frac{F_1/F}{1+\beta} \tag{5.1-8}$$

$$K_2 = \frac{F_2/F}{1+\beta} \tag{5.1-9}$$

Gains K_1 and K_2 show the effects of inlet temperatures T_1 and T_2 on the outlet temperature. For example, a change in T_1 will have a small effect on T_o if the inlet flow rate F_1 is small compared to overall flow F.

$$K_3 = \frac{\beta}{1+\beta} \tag{5.1-10}$$

Gain K_3 shows the effect of changes in the temperature T_c of the condensing vapor. For high heat transfer capability, β is large, and gain K_3 approaches unity.

Our system model of the heated tank is finally written

$$\tau \frac{dT_o'}{dt} + T_o' = K_1 T_1' + K_2 T_2' + K_3 T_c' \tag{5.1-11}$$

which shows a first-order system with three inputs. As is our custom, we will take the initial condition on T'_o as zero.

5.2 solving by laplace transform

Solution by Laplace transform is straightforward:

$$\begin{aligned}
&L\left\{\tau \frac{dT_o'}{dt} + T_o'\right\} = L\left\{K_1 T_1' + K_2 T_2' + K_3 T_c'\right\} \\
&\tau L\left\{\frac{dT_o'}{dt}\right\} + L\left\{T_o'\right\} = K_1 L\left\{T_1'\right\} + K_2 L\left\{T_2'\right\} + K_3 L\left\{T_c'\right\} \\
&\tau\left(sT_o'(s) - T_o'(0)\right) + T_o'(s) = K_1 T_1'(s) + K_2 T_2'(s) + K_3 T_c'(s) \\
&T_o'(s) = \frac{K_1}{\tau s + 1} T_1'(s) + \frac{K_2}{\tau s + 1} T_2'(s) + \frac{K_3}{\tau s + 1} T_c'(s)
\end{aligned} \tag{5.2-1}$$

Output T'_o is related to three inputs, each through a first-order transfer function.

5.3 response of system to step disturbance

Suppose the tank is disturbed by a step change ΔT_1 in temperature T_1. We have studied first-order systems, so we already know what the first-order step response looks like: at the time of disturbance t_d, the output T'_o will depart from its initial zero value toward an ultimate value equal to the product of gain K_1 and the step ΔT_1. The time required depends on the magnitude of time constant τ: when time equal to one time constant has passed (i.e., $t = t_d + \tau$) the outlet temperature will be about 63% of its way toward the long-term value.

However, doing the formalities for two step disturbances and one steady input,

$$T_1' = \Delta T_1 U(t - t_{d1}) \qquad T_2' = \Delta T_2 U(t - t_{d2}) \qquad T_c' = 0 \tag{5.3-1}$$

We take Laplace transforms of these input variables:

$$T_1'(s) = \frac{\Delta T_1}{s} e^{-t_{d1}s} \qquad T_2'(s) = \frac{\Delta T_2}{s} e^{-t_{d2}s} \qquad T_c'(s) = 0 \tag{5.3-2}$$

Then we substitute (5.3-2) into the system model (5.2-1).

$$T_o'(s) = \frac{K_1}{\tau s+1}\frac{\Delta T_1}{s} e^{-t_{d1}s} + \frac{K_2}{\tau s+1}\frac{\Delta T_2}{s} e^{-t_{d2}s} + \frac{K_3}{\tau s+1} 0 \tag{5.3-3}$$

We must invert each term; this is most easily done by processing the polynomial first and then applying the time delay. Thus

$$L^{-1}\left\{\frac{1}{\tau s+1}\frac{1}{s}\right\} = 1 - e^{-t/\tau} \tag{5.3-4}$$

$$L^{-1}\left\{\left(\frac{1}{\tau s+1}\frac{1}{s}\right)e^{-t_{d1}s}\right\} = U(t - t_{d1})\left(1 - e^{-(t-t_{d1})/\tau}\right) \tag{5.3-5}$$

and finally

$$T_o^* = K_1 \Delta T_1 U(t - t_{d1})\left(1 - e^{-(t-t_{d1})/\tau}\right) + K_2 \Delta T_2 U(t - t_{d2})\left(1 - e^{-(t-t_{d2})/\tau}\right) \tag{5.3-6}$$

Figure 5.3-1 shows a sample calculation for opposing input disturbances. Notice that the gains K_1 and K_2 are less than unity. This is because we dilute each disturbance with other streams: streams of matter (two input flows) and energy (through the heat transfer surface). From the plot, can you tell which gain is greater?

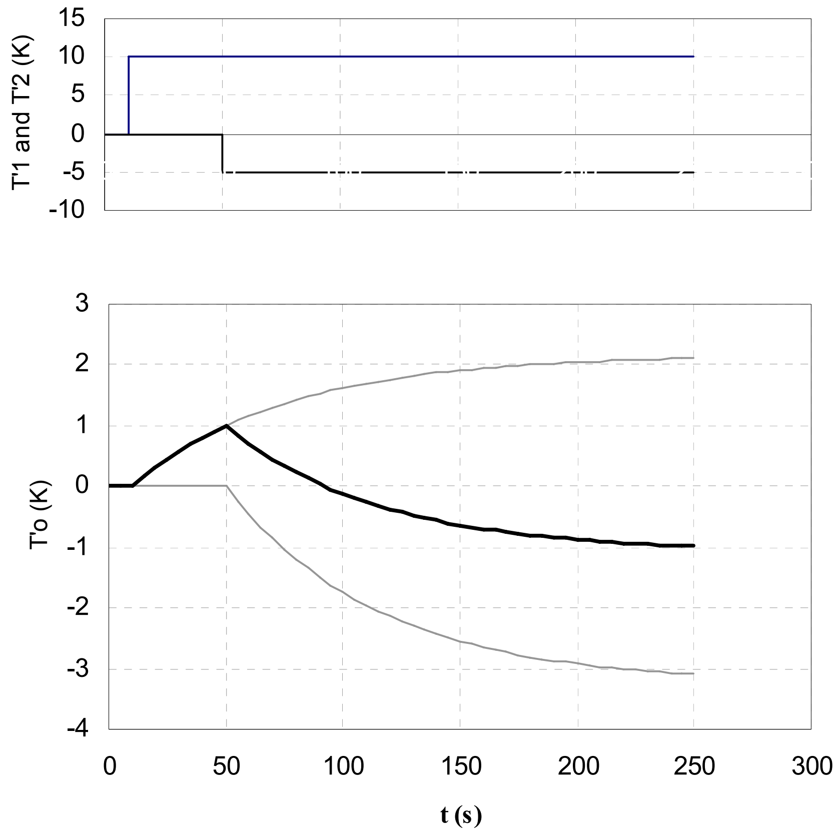

Figure 5.3-1: Response of outlet temperature to steps in two inlets

5.4 stability of the heated tank

System model (5.2-1) has one negative pole, $s = -\tau^{-1}$, because the time constant is a positive quantity. Hence, the system is stable to all bounded inputs. Therefore, a blip in the inlet temperature will not set off wild temperature excursions at the outlet.

5.5 numerical solution of ODEs

Analytical solutions to equations are desirable, but many useful equations simply cannot be solved. When we "solve an ODE numerically" we execute a set of calculations that results in a "numerical solution": in effect, a table of numbers. One column in that table contains values of the independent variable, and the other column holds associated values of the dependent variable. A table of numbers lacks the economy of presentation and conceptual insight offered by an analytical expression. However, a table of numbers is much better than no solution, and it can certainly be plotted.

Matlab offers a suite of functions for solving differential equations. For example, the following file contains code to produce a plot indistinguishable from that of the analytical solution in Figure 5.3-1.

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```
function heated_tank (tauR, beta, F1frac)
% program to solve Eqn (5.1-11)
% the temperature of the heated tank is disturbed by the temperatures of
% the inlet streams and the condensing temperature of the vapor in the
% heat exchanger bundle

% INPUT variables
% tauR      residence time in seconds
% beta      heat transfer significance parameter
% F1frac    fraction of flow in stream 1

% OUTPUT variables
% To        the deviation in outlet temperature is plotted

% NOTE: all system variables are in deviation form

% define equation parameters
tau = tauR/(1+beta) ; % time constant in seconds
K1 = F1frac/(1+beta) ; % stream 1 gain
K2 = (1 - F1frac)/(1+beta) ; % stream 2 gain
K3 = beta/(1+beta) ; % heat exchange gain

% define the disturbances - first interval
    tspan = [0, 10] ; % set the time interval
    Toinit = 0 ; % start out at reference condition
    T1 = 0; T2 = 0; Tc = 0; % no disturbances

    % integrate the equation
    [t,To] = ode45(@hot_tank,tspan,Toinit,[],tau,K1,K2,K3,T1,T2,Tc) ;

    % plot the solution
    plot (t,To)
    hold on % allow plot to be updated with further plotting

% define the disturbances - second interval
    tspan = [10, 50] ; % set the time interval
    Toinit = To(size(To,1)); % start out at most recent value
    T1 = 10; T2 = 0; Tc = 0; % introduce step in T1

    % integrate the equation
    [t,To] = ode45(@hot_tank,tspan,Toinit,[],tau,K1,K2,K3,T1,T2,Tc) ;

    % plot the solution
    plot (t,To)

% define the disturbances - third interval
    tspan = [50, 250] ; % set the time interval
    Toinit = To(size(To,1));  % start out at most recent value
    T1 = 10; T2 = -5; Tc = 0; % introduce step in T2

    % integrate the equation
    [t,To] = ode45(@hot_tank,tspan,Toinit,[],tau,K1,K2,K3,T1,T2,Tc) ;

    % plot the solution
    plot (t,To)
```

```
    hold off % turn off the plot hold

end % heated_tank

% define the differential equation in a subfunction
function dTodt = hot_tank(t,To,tau,K1,K2,K3,T1,T2,Tc)
dTodt = (K1*T1 + K2*T2 + K3*Tc - To)/tau ;
end
```

The Matlab function `ode45` is one of several routines available for integrating ordinary differential equations. The output arguments are a column vector for time `t` and another for temperature deviation `To`. The input argument `@hot_tank` directs `ode45` to the subfunction in which the equation is defined. Notice the way in which `hot_tank` is written: the equation is arranged to calculate the derivative of output variable `To` as a function of `To` and the input variables. `ode45` calls `hot_tank` repeatedly as it marches through the time interval denoted by the row vector `tspan`.

5.6 scaled variables versus deviation variables

We introduced deviation variables so that any non-zero variable - positive or negative - would be seen as a departure from the ideal reference condition. Deviation variables also allowed us to define and use transfer functions in expressing our system models through Laplace transforms.

We must, however, finally read real instruments and control real processes. For these purposes, it is common to represent sensor readings, controller outputs, and valve-stem positions on a scale of 0 to 100. Such scaled variables obscure actual values, but immediately reveal context. That is, a visitor to a control room would not know the significance of a particular tank level reading of 2.6 m, but could interpret 96% easily. An everyday example of a sensor that presents a scaled variable is the fuel gauge in an automobile.

To scale physical variable y, for example, we identify the range in which we expect it to vary: from y_{min} to y_{max}. We then subtract some bias value from y and divide the difference by the range:

$$y^* \equiv \frac{y - y_b}{y_{max} - y_{min}} 100\% \qquad (5.6\text{-}1)$$

If the bias value y_b is set to the minimum y_{min}, then y^* varies between 0 and 100%. This is the typical control room presentation. If instead the bias value is set to the reference value y_r, then y^* varies from

$$\frac{y_{min} - y_r}{y_{max} - y_{min}} 100\% \le y^* \le \frac{y_{max} - y_r}{y_{max} - y_{min}} 100\% \qquad (5.6\text{-}2)$$

The scaled range is still 100% wide, but includes both positive and negative regions, depending on where y_r lies between y_{min} and y_{max}. Of course, y_b may be set to any arbitrary value between the limits, but y_{min} and y_r are generally the most useful.

We will use primes (′) to denote deviation variables and asterisks (*) for scaled variables. Unadorned variables will be presumed to be physical. To convert a deviation variable y′ (from an analytical solution, e.g.) for presentation as a scaled variable y^*, the definitions are combined:

$$y^* \equiv \frac{\left(y' + y_r\right) - y_b}{y_{max} - y_{min}} 100\% \qquad (5.6\text{-}3)$$

For example, Figure 5.6-1 shows a temperature trace expressed in physical, deviation, and scaled form. Because these are linear transformations, the basic character of the variable is unchanged.

5.7 just when I was getting accustomed to deviation variables!

We will tend to use deviation variables for analytic solutions and derivations because of the convenience of zero initial conditions. We will use scaled variables for our numerical work, in which each time step moves the solution to a new value from the "initial condition" of the previous step.

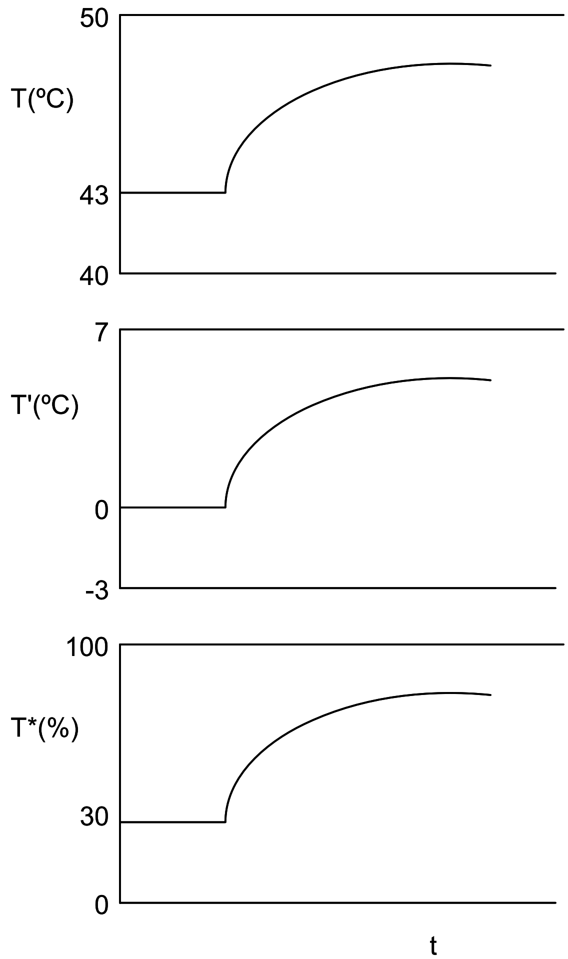

Figure 5.6-1 Expressing the variable in physical, deviation, and scaled forms

CONTROL SCHEME

5.8 step 1 - specify a control objective for the process

Our control objective is to maintain the outlet temperature T_o at a constant value. In particular, we prefer not to have offset in response to step-like inputs. This means we must do something besides proportional mode.

5.9 step 2 - assign variables in the dynamic system

The controlled variable is clearly T_o. The inlet temperatures T_1 and T_2 are disturbance variables.

By the model, we are left with the condensing vapor temperature as the manipulated variable. But what sort of valve adjusts temperature? We will discuss this below when we select equipment.

5.10 step 3 - PI (proportional-integral) control

We introduced proportional control as an intuitively appealing mechanism - the response increases with the severity of the error. However, proportional control suffers from offset. To counter this defect, we introduce a second mode of control, called integral, to complement the proportional behavior.

$$x^*_{co} - x^*_{co,b} = K^*_c\left(\varepsilon^* + \frac{1}{T_I}\int_0^t \varepsilon^* dt\right) \qquad (5.10\text{-}1)$$

where x^*_{co} is the controller output and the controlled variable error is

$$\varepsilon^* = y^*_{sp} - y^* \qquad (5.10\text{-}2)$$

Before we discuss the integral mode, let us understand why we have written the controller algorithm in scaled variables. The controller is not specific to a process; it merely produces a response according to the error it receives. Hence it is most reasonable to build it so that error and response are expressed as percentages of a range; whatever y may be as a physical variable, in Equation (5.10-2) the scaled controlled variable y^* is subtracted from y^*_{sp}, which is the preferred position on the scale. The resulting error ε^* is processed by the controller in Equation (5.10-1) to generate a scaled output x^*_{co}. The bias output $x^*_{co,b}$ is the controller's resting state, achieved when there has been no error.

Integral mode integrates the error, so that the controller output x^*_{co}, which drives the manipulated variable in the loop, increases with the *persistence* of error ε^*, in addition to its severity. The influence of the integral mode is set by the magnitude of the integral time T_I. In the special case of a constant error input to the controller, T_I is the time in which the controller output doubles. Thus decreasing T_I strengthens the controller response. Very large T_I disables the integral mode, leaving a proportional controller. The dimensionless controller gain K^*_c acts on both the proportional and integral modes.

As an example, let us test a controller in isolation, so that we supply a controlled variable independently, and the controller output has no effect. Suppose we set K^*_c to 1 and T_I to 1 minute. Suppose the set point is 40% and the bias output 50%. Initially, there is no error, and so the controller output is 50%. Figure 5.10-1 shows the controller response to a step increase in the controlled variable from 40 to 60%. Because the error suddenly becomes -20%, the proportional mode calls for the output to change by -20% (gain of +1). Thus x^*_{co} becomes 30%. Over the next minute, there is no change in the controlled variable. Hence the integral mode calls for more controller output. In 1 minute, x^*_{co} decreases by another 20% and so reaches 10%.

At one minute, we contrive to return the controlled variable to set point, so that the error is again zero. The proportional mode then ceases to call for output. However, the integral mode still remembers the earlier error, and so the output returns to 30%, not the original 50%. We will see later that when the controller is placed in a feedback loop, not isolated as we have used it here, the integral mode acts to eliminate offset.

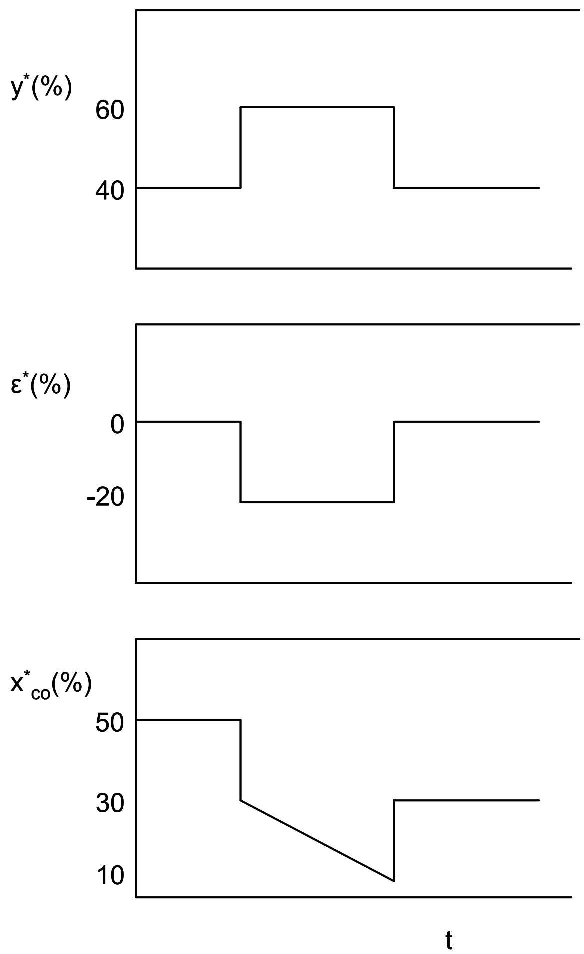

Figure 5.10-1: response of isolated PI controller to an input pulse

For our analytical work, we will want to express the controller algorithm in deviation variables. We proceed by substituting from definition (5.6-1) into the algorithm (5.10-1) and (5.10-2).

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$$
\begin{aligned}
\varepsilon^{*} &= \frac{y_{sp} - y_{min}}{y_{max} - y_{min}}100\% - \frac{y - y_{min}}{y_{max} - y_{min}}100\% \\
&= \left(y_{sp} - y\right)\frac{100\%}{y_{max} - y_{min}} \\
&= \left(y'_{sp} - y'\right)\frac{100\%}{y_{max} - y_{min}} \\
&= \varepsilon\frac{100\%}{y_{max} - y_{min}}
\end{aligned}
\tag{5.10-3}
$$

We can see that ε^{*} represents the physical variable error ε multiplied by a ratio of scaling ranges. Continuing with the controller output

$$
\begin{aligned}
x^{*}_{co} - x^{*}_{co,b} &= \frac{x_{co} - 0\%}{100 - 0\%}100\% - \frac{x_{co,b} - 0\%}{100 - 0\%}100\% \\
&= x_{co} - x_{co,b} \\
&= x_{co} - x_{co,r} \\
&= x'_{co}
\end{aligned}
\tag{5.10-4}
$$

In (5.10-4) we recognize that the controller bias is most reasonably thought of as its value at the reference condition. Also, because controller output ranges from 0 to 100%, its "physical" value is identical to its scaled value. Rewriting algorithm (5.10-1)

$$
\begin{aligned}
x'_{co} &= K^{*}_{c}\frac{100\%}{\Delta y}\left(\varepsilon + \frac{1}{T_I}\int_0^t \varepsilon dt\right) \\
&= K_c\left(\varepsilon + \frac{1}{T_I}\int_0^t \varepsilon dt\right)
\end{aligned}
\tag{5.10-5}
$$

The dimensionless controller gain K^{*}_{c} (the setting that would actually be found "on" the controller itself) is multiplied by the ratio 100% Δy^{-1} to produce a dimensional quantity K_c. K_c converts the dimensions of the error ε to the % units of controller output x_{co}. If the error ε were expressed in the units of a physical variable (a liquid level, for example), Δy would perhaps be some number of centimeters. If error were instead expressed in terms of the output of a signal transducer on the measuring instrument, Δy might be in volts or milliamps.

The Laplace transform of controller algorithm (5.10-5) is

$$x'_{co}(s) = K_c\left(\varepsilon(s) + \frac{1}{T_I s}\varepsilon(s)\right) \qquad (5.10\text{-}6)$$

5.11 step 4 - choose set points and limits

The heated blending tank might require an occasional change of set point, depending on product grade, time of year, other process conditions, etc. At present we are presuming that the tank is part of a continuous process, so that its normal operating mode is to maintain temperature at the set point. Such a tank could also be used in a batch process, however. In this case, the set point might be an active function of time, according to the recipe of the batch.

Limits placed on temperature can be both high and low, depending on the process. Reasons for imposing high limits are often undesirable chemical changes: polymerization, product degradation, fouling, side reactions. Both high and low limits may be imposed to avoid phase changes: boiling and freezing for liquids.

We institute regulatory control, using, e.g., proportional-integral controllers, to keep controlled variables within acceptable operating limits. However, when there are safety limits to be enforced, regulatory control may be superseded by a safety control system. Thus a reactor temperature may normally be regulated within operating limits, but some higher value will trigger an audible alarm in the control room, and some yet higher value will initiate emergency response or shutdown procedures that override normal regulatory control.

EQUIPMENT

5.12 the sensor in the feedback control loop

Temperature may be measured with a variety of instruments that respond to temperature with an electrical signal, including thermocouples, thermistors, RTDs (resistance thermometry devices), etc. In this section, we address both the static (calibration) and dynamic (time response) characteristics of temperature sensors, with reference to our heated tank example.

Calibration of the sensor is determining the relationship between the actual quantity of interest (the temperature at some location in the fluid) and the output given by the sensor (which can be a voltage, a current in a circuit, a digital representation, etc., depending on the instrument). When we speak of a sensor, we usually refer to both the sensing element (such as the bimetallic junction of a thermocouple) and signal conditioning electronics. It is this latter component that produces a linear relationship between

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temperature and sensor output, even though the behavior of the sensing element itself may be nonlinear.

Thus we relate the physical temperature T_o and its sensor reading T_s by

$$T_s = K_s T_o + b_s \tag{5.12-1}$$

In a handheld digital thermometer, the electronics are adjusted so that gain K_s is unity and bias b_s is zero: 26ºC produces a reading of 26ºC. In a control loop, however, we are more likely to have T_o produce an electric current that ranges over 4 to 20 mA, where these limits correspond to the expected range of temperature variation. Current loops are a good way to transmit signals over the sorts of distances that separate operating processes from their control rooms.

The sensor range is adjusted by varying K_s and b_s. For example, suppose that we wish to follow T_o over the range 50 to 100ºC. Then

$$\begin{aligned} (4)\,\mathrm{mA} &= K_s (50)^\circ \mathrm{C} + b_s \\ (20)\,\mathrm{mA} &= K_s (100)^\circ \mathrm{C} + b_s \\ \Rightarrow K_s &= (0.32)\,\mathrm{mA\,K^{-1}} \qquad b_s = (-12)\,\mathrm{mA} \end{aligned} \tag{5.12-2}$$

We express the sensor calibration in deviation variables by subtracting the reference state from (5.12-1). Suppose we wish to use 75ºC as a reference operating condition. At the reference, the sensor output will be 12 mA.

$$T_s' = K_s T_o' \tag{5.12-3}$$

We obtain a 0 to 100% range by taking the scaled variable reference as the minimum temperature.

$$T_s^* = \frac{T_s - (4)\mathrm{mA}}{(20-4)\mathrm{mA}} 100\% = \frac{T_o - (50)^\circ \mathrm{C}}{(100-50)^\circ \mathrm{C}} 100\% \tag{5.12-4}$$

From (5.12-4), we see that the scaled sensor output may be defined in terms of the sensor reading or the controlled variable itself. A temperature of 75ºC causes a sensor output of 12 mA, or 50% of range.

We now consider the dynamic response of our sensor. Suppose we place the sensor in fluid at 50ºC and allow it to equilibrate, so that its output is 4 mA. We now move the sensor suddenly to another fluid at 100ºC; how quickly does the sensor respond to this step input? How long until the output becomes 20 mA? Classic textbook treatments of thermocouples and thermometers indicate that the sensor response is first-order. Hence, we may write the sensor transfer function as

$$T_s'(s) = \frac{K_s}{\tau_s s + 1} T_o'(s) \qquad (5.12\text{-}5)$$

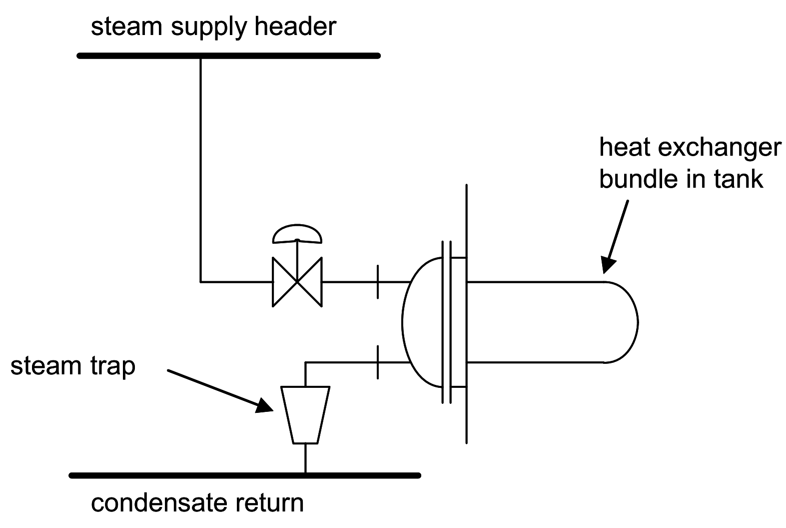
In (5.12-5) K_s is the sensor gain that was determined by calibration in (5.12-2) and (5.12-3). The sensor has a time constant τ_s that depends on, e.g., the mass of the sensor element and the rate of heat transfer to the sensor. It will follow a step input in T'_o with the familiar first-order exponential trace.

However, we plan to immerse the sensor in a large stirred tank. Therefore, we are unlikely to encounter step changes in T_o. In fact, it is often the case that the sensor time constant τ_s is small in comparison to the tank time constant τ, given in (5.1-7). This implies that T_s will keep up with changes in T_o, so that we can simplify (5.12-5) to a pure gain process.

$$T_s'(s) = K_s T_o'(s) \qquad (5.12\text{-}6)$$

5.13 the valve in the feedback control loop

Physically, the controller output dictates the opening of a valve that admits the heating fluid to the heat exchanger. The figure shows a steam supply header, a line to the heat exchanger, and a steam trap at the exit, which delivers condensate to the condensate return header.



Viewing the control valve as a dynamic system, we think of the valve converting the controller output (measured as %out) to the temperature of the condensing steam (measured in degrees). This is not quite as farfetched as it may sound:

- the controller output (between 0 and 100%) specifies the position of the valve stem (between closed and open).
- The valve opening varies the resistance presented to the flow.
- Resistance relates any flow through the valve to the pressure drop required by that flow.
- The pressure drop across the valve relates the steam supply pressure to the pressure at which the steam condenses in the heat exchanger.
- The prevailing pressure in the heat exchanger determines the condensing temperature.
- The condensing temperature determines the rate of heat transfer from vapor to tank.
- The heat transfer rate determines the rate at which steam condenses to supply the heat.

Thus the flow admitted by the valve is the flow that is able to condense at a temperature high enough to transfer that heat of condensation to the liquid in the tank. In short, we open the valve to supply more heat.

Let us not become lost among momentum equation, vapor pressure relationship, empirical flow resistance relationships, etc., which we invoked above. Our purpose is to describe the action of the controller upon the condensing temperature, and so we need a gain (from steady-state relationships) and dynamic description (from the rate processes). Will a simple first-order description be sufficient?

$$T_c'(s) = \frac{K_v}{\tau_v s + 1} x_{co}'(s) \qquad (5.13\text{-}1)$$

We will address this question more thoroughly in Lesson 6. For the present, while not dismissing our skepticism, we will accept (5.13-1). That is, we presume that <u>over some range of operating conditions</u>, at least, the change in condensing temperature is directly proportional to a change in controller output. The underlined phrase is a key one: our linear (i.e., constant gain) model is an approximation, and part of our engineering job is to determine how far our model can be trusted.

Regarding dynamic response, the first-order response of T_c to x_{co} is also an approximation. In Lesson 4 we used this description of our valve, and we saw that our second-order process combined with the valve to produce a third-order closed loop. In this lesson, however, we have other topics to explore, so we will presume that the characteristic time τ_v for changing the condensing temperature is much smaller than the mixing tank time constant τ, so that

$$T_c'(s) = K_v x_{co}'(s) \qquad (5.13\text{-}2)$$

We have said, then, that we expect the dynamic response of the closed loop to depend primarily on the dynamic characteristics of the process, and very little on the characteristics of sensor (Section 5.12) and valve (Section 5.13).

5.14 numerical controller calculations

In Lessons 3 and 4, we have expressed controller behavior in mathematical terms and predicted the closed loop behavior by solving process and controller equations simultaneously. We posit "step disturbance" and then make a plot of what the equations say. This dynamic system representation is useful if it can help us manage real equipment.

We understand that our process is actually a tank, but what does the controller look like? For many years, the controller was a physical box that manipulated air flow with bellows and dampers; its output was an air pressure that positioned a valve stem. Coughanowr and Koppel (chap.22) describe such mechanisms.

Now, the controller is most often a program that runs on a microprocessor. The program input is numbers that represent physical signals from the sensor. The algorithm (5.10-1) is computed numerically. The output numbers are fed to a transducer that makes physical adjustments to a valve.

Whereas our analytical solutions represent continuous connection between controller and process, the computer program controller samples the process values at intervals. Marlin (chap.11) discusses the effect of sampling frequency on controller performance.

A very simple controller program outline is given below:

initialization
- set up arrays to hold variables
- set controller parameters (Kc*, TI)

loop for automatic control
- interrogate sensor to determine controlled variable at t_now
- use comparator to determine set point and calculate error
- use controller algorithm to calculate controller output at t_now
- administer controller output to process
- update monitor plot with values at t_now
- wait for time interval Δt to pass
- set new value of t_now and return to beginning of loop

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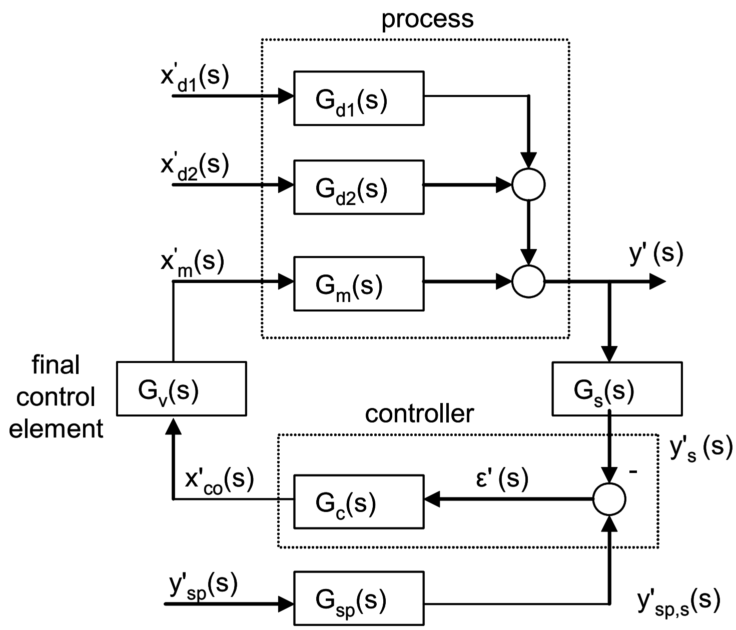
In this course, we will embellish this scheme with coding to simulate the process, as well, so that we can make numerical simulations of processes under control.

CLOSED LOOP BEHAVIOR

5.15 closed loop transfer functions

We recall the general feedback block diagram introduced in Lesson 4. Here we modify it to show our two disturbance inputs, but the control architecture is unchanged.



From the diagram, our general closed loop model is

$$y_c'(s) = \frac{G_{d1}}{\left(1 + G_m G_v G_c G_s\right)} x_{d1}'(s) + \frac{G_{d2}}{\left(1 + G_m G_v G_c G_s\right)} x_{d2}'(s) + \frac{G_m G_v G_c G_{sp}}{\left(1 + G_m G_v G_c G_s\right)} y_{sp}'(s) \quad (5.15\text{-}1)$$

We apply (5.15-1) to our heated tank by inserting the particular transfer functions from Sections 5.2, 5.12, 5.13, and 5.14. Thus the disturbance transfer function is

$$\frac{T_o'(s)}{T_1'(s)} = \frac{\dfrac{K_1 T_I}{K_s K_c K_v K_3} s}{\dfrac{\tau T_I}{K_s K_c K_v K_3} s^2 + T_I \dfrac{1 + K_s K_c K_v K_3}{K_s K_c K_v K_3} s + 1} \quad (5.15\text{-}2)$$

(and similarly for disturbance T_2) and the set point transfer function is

$$\frac{T_o^{'}(s)}{T_{sp}^{'}(s)} = \frac{T_I s + 1}{\frac{\tau T_I}{K_s K_c K_v K_3} s^2 + T_I \frac{1 + K_s K_c K_v K_3}{K_s K_c K_v K_3} s + 1} \tag{5.15-3}$$

Adding integral control to a first order process has resulted in a closed loop with second-order dynamics.

5.16 closed-loop behavior - set point step response

Responses to disturbance inputs and set point changes will depend on the poles of (5.15-3). These are

$$s_1, s_2 = \frac{-1 - K_s K_c K_v K_3 \pm \sqrt{1 + \left(2 - \frac{4\tau}{T_I}\right) K_s K_c K_v K_3 + \left(K_s K_c K_v K_3\right)^2}}{2\tau} \tag{5.16-1}$$

We observe that the poles could be complex, so that the closed loop response could be oscillatory. The tendency toward a negative square root, and thus oscillation, is exacerbated by reducing the integral time T_I. We also observe that the real part of the poles is negative, indicating a stable system.

We illustrate set point step response for real poles, where the set point is changed by magnitude ΔT:

$$T_o^{'} = \Delta T \left[1 + \frac{\frac{-1}{s_1} - T_I}{\frac{-1}{s_2} - \frac{-1}{s_1}} e^{s_1 t} - \frac{\frac{-1}{s_2} - T_I}{\frac{-1}{s_2} - \frac{-1}{s_1}} e^{s_2 t} \right] \tag{5.16-2}$$

The response is written in terms of poles s_1 and s_2. Because they are negative, the two exponential terms decay in time, leaving the long-term change in set point as ΔT. Thus we requested that the tank temperature change by ΔT, and the tank temperature changed by ΔT. There is no offset - this is the contribution of the integral mode of control.

5.17 second order systems

In Lesson 4, we described the two tanks in series as an overdamped second-order system; now our heated tank with integral-mode control is also second order, but perhaps not overdamped. We defer further exploration of our closed loop behavior until we learn more about the properties of second-order systems. Thus we consider

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$$\alpha \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + y(t) = Kx(t) + M\frac{dx}{dt}$$
$$y(0) = \left.\frac{dy}{dt}\right|_{t=0} = 0 \qquad (5.17\text{-}1)$$

As before, we will assume that all initial conditions on response variable y(t) are zero, so that the system is initially at steady state, and will be driven only by disturbance x(t). Parameter K has dimensions of y divided by x; it is the steady-state gain. Parameter M measures the sensitivity of the system to the rate of change of the disturbance x(t); it has the dimensions of K multiplied by time. Both K and M may be positive or negative, as may α and β.

We solve (5.17-1) by Laplace transform.

$$\frac{y(s)}{x(s)} = \frac{Ms + K}{\alpha s^2 + \beta s + 1} \qquad (5.17\text{-}2)$$

The behavior of the solution depends fundamentally on the poles of (5.17-2). From the properties of the quadratic equation we recall that the poles are real for

$$\alpha < \frac{\beta^2}{4} \qquad (5.17\text{-}3)$$

A map relating system stability to coefficients α and β is given in Figure 5.17-1.

Lesson 5: Heated Tank

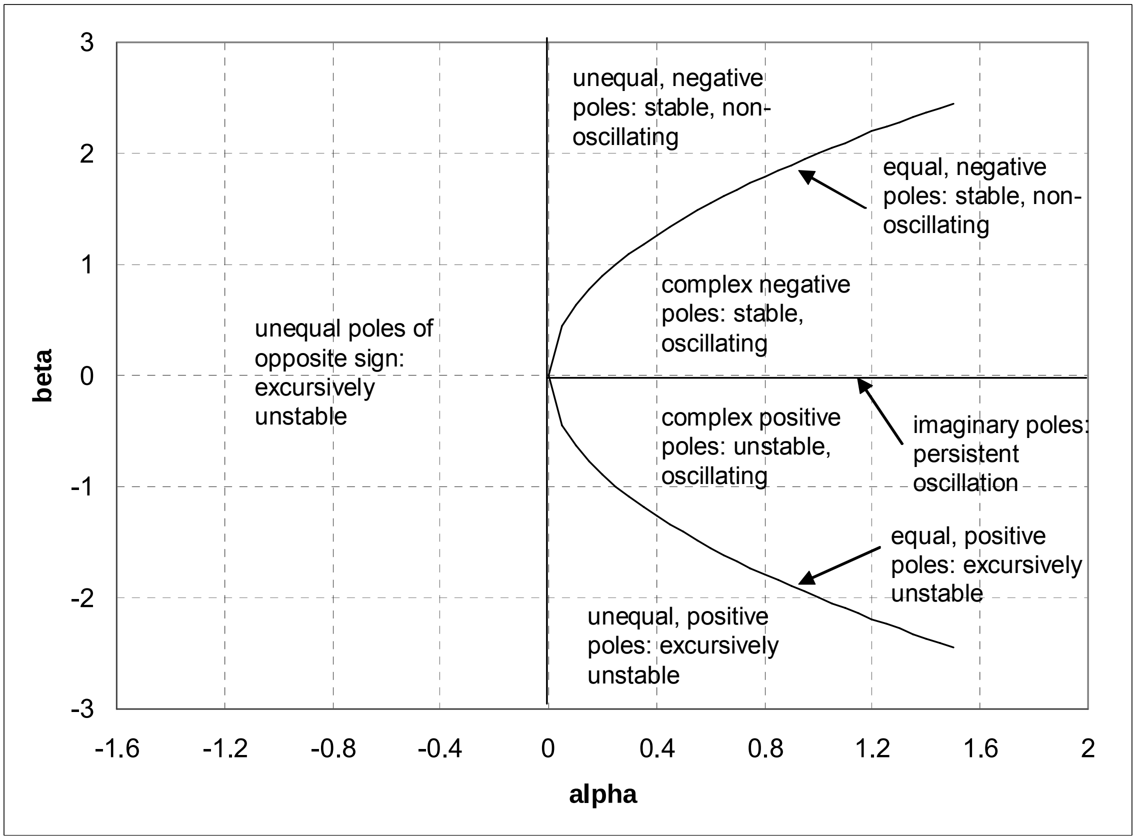

Figure 5.17-1. Second-order system stability related to equation coefficients

With real poles, it is convenient to factor the denominator, and thus express the second-order system in terms of two first-order systems in which the characteristic times τ_1 and τ_2 may be positive or negative.

$$\tau_1, \tau_2 = \frac{\beta \pm \sqrt{\beta^2 - 4\alpha}}{2} \qquad (5.17\text{-}4)$$

We compare side-by-side second-order and first-order systems:

Table 5.17-1. Second-order system (τ_1, τ_2 form) – real poles

	second-order	first-order
equation	$\tau_1\tau_2 \frac{d^2y}{dt^2} + (\tau_1 + \tau_2)\frac{dy}{dt} + y(t) = Kx(t) + M\frac{dx}{dt}$ $y(0) = \left.\frac{dy}{dt}\right\|_{t=0} = 0$	$\tau\frac{dy}{dt} + y(t) = Kx(t)$ $y(0) = 0$
transfer function	$\frac{y(s)}{x(s)} = \frac{K + Ms}{(\tau_1 s + 1)(\tau_2 s + 1)}$	$\frac{y(s)}{x(s)} = \frac{K}{\tau s + 1}$
Poles	$s = -\frac{1}{\tau_1}, -\frac{1}{\tau_2}$	$s = -\frac{1}{\tau}$

Zeroes	$s = \frac{-K}{M}$	none

The system will be stable if both τ_1 and τ_2 are positive, as they were for the mixing tanks of Lesson 4. A second-order step response is

$$y(t) = AKU(t-t_d)\left[1-\left(1-\frac{M}{K\tau_1}\right)\frac{\tau_1}{\tau_1-\tau_2}e^{-(t-t_d)/\tau_1}+\left(1-\frac{M}{K\tau_2}\right)\frac{\tau_2}{\tau_1-\tau_2}e^{-(t-t_d)/\tau_2}\right] \quad (5.17\text{-}5)$$

When α, the coefficient of the second derivative, is positive (right-hand-side of Figure 5.17-1), it is customary to express the second-order system in an alternative way:

Table 5.17-2. Second-order system (τ, ξ form) – complex poles

	second-order	first-order
equation	$\tau^2\frac{d^2y}{dt^2}+2\xi\tau\frac{dy}{dt}+y(t)=M\frac{dx}{dt}+Kx(t)$ $y(0)=\left.\frac{dy}{dt}\right\|_{t=0}=0$	$\tau\frac{dy}{dt}+y(t)=Kx(t)$ $y(0)=0$
transfer function	$\frac{y(s)}{x(s)}=\frac{Ms+K}{\tau^2s^2+2\xi\tau s+1}$	$\frac{y(s)}{x(s)}=\frac{K}{\tau s+1}$
poles	$s=-\frac{1}{\tau}\left(\xi\pm\sqrt{\xi^2-1}\right)$	$s=-\frac{1}{\tau}$
zeroes	$s=\frac{-K}{M}$	none

Relationships between the characteristic times τ_1 and τ_2 and the alternative parameters τ and ξ are

$$\tau_1=\frac{\tau}{\xi-\sqrt{\xi^2-1}}=\tau\left(\xi+\sqrt{\xi^2-1}\right) \quad (5.17\text{-}6)$$

$$\tau_2=\frac{\tau}{\xi+\sqrt{\xi^2-1}}=\tau\left(\xi-\sqrt{\xi^2-1}\right) \quad (5.17\text{-}7)$$

$$\tau=\sqrt{\tau_1\tau_2} \quad (5.17\text{-}8)$$

$$\xi=\frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}} \quad (5.17\text{-}9)$$

$$\tau_1+\tau_2=2\tau\xi \quad (5.17\text{-}10)$$

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$$\tau_1 - \tau_2 = 2\tau\sqrt{\xi^2 - 1} \tag{5.17-11}$$

As in the first-order system, τ has the dimension of time and represents a characteristic time of the system. We will take it to be positive; dimensionless ξ in the second-order system, however, we will allow to be positive or negative. Parameter ξ is called the damping coefficient, and the character of the response depends markedly on its value, as we will explore in the next few sections.

5.18 overdamped systems: $\xi > 1$

The poles are real, negative, and unequal. We have seen such a system in Lesson 4, a system of unequal tanks in series. The overdamped response to an input step of magnitude A is

$$y(t) = AKU(t - t_d)\left[1 - e^{-\xi(t-t_d)/\tau}\left\{\begin{aligned}&\left(1 - \frac{M}{K\tau}\left(\xi - \sqrt{\xi^2-1}\right)\right)\frac{\xi + \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}\,e^{\sqrt{\xi^2-1}(t-t_d)/\tau}\\ &-\left(1 - \frac{M}{K\tau}\left(\xi + \sqrt{\xi^2-1}\right)\right)\frac{\xi - \sqrt{\xi^2-1}}{2\sqrt{\xi^2-1}}\,e^{-\sqrt{\xi^2-1}(t-t_d)/\tau}\end{aligned}\right\}\right] \tag{5.18-1}$$

Figure 5.18-1 shows step responses for various ξ, with $M = 0$. In contrast to the first-order response, the trace has a sigmoid shape: a slow start and an inflection point. Larger ξ make the response more sluggish.

5.19 critically damped systems: $\xi = 1$

For this special condition, the poles are real, negative, and equal. This is the case for two identical tanks in series. The step response is

$$y(t) = AKU(t - t_d)\left[1 - e^{-(t-t_d)/\tau}\left\{1 + \left(1 - \frac{M}{K\tau}\right)\frac{(t - t_d)}{\tau}\right\}\right] \tag{5.19-1}$$

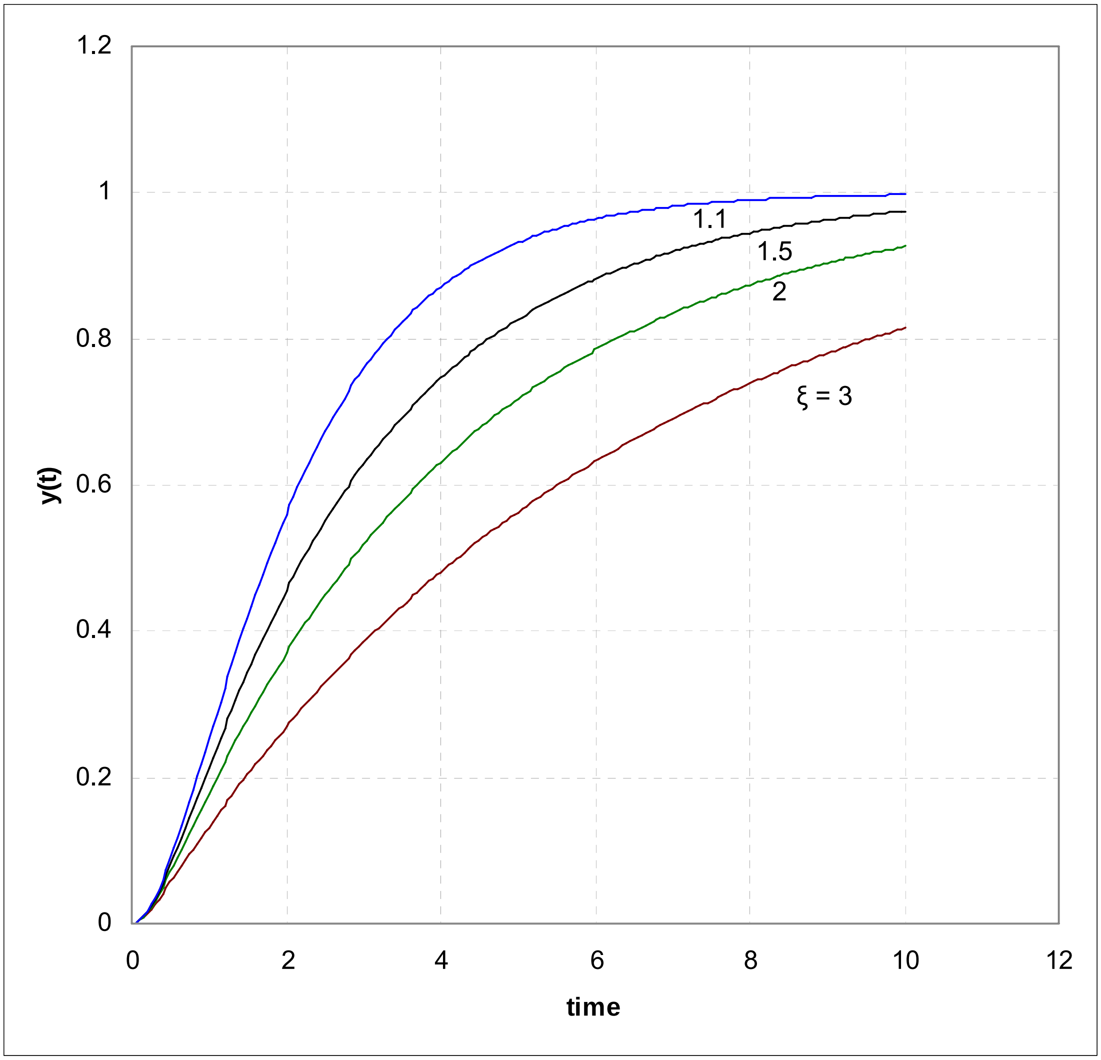


Figure 5.18-1. Step response for overdamped second-order system with M = 0.

5.20 underdamped systems: $\xi < 1$

The poles are complex conjugates, and the system will oscillate even for a non-periodic disturbance. The overdamped response (5.18-1) may be modified by substituting

$$\sqrt{\xi^2 - 1} = \sqrt{-\left(1-\xi^2\right)} = j\sqrt{1-\xi^2} \tag{5.20-1}$$

and using Euler's relation to find

$$y(t) = AKU\left(t-t_d\right)\left[1 - e^{-\xi(t-t_d)/\tau}\left\{\cos\sqrt{1-\xi^2}\,\frac{\left(t-t_d\right)}{\tau} + \frac{\xi - \frac{M}{K\tau}}{\sqrt{1-\xi^2}}\sin\sqrt{1-\xi^2}\,\frac{\left(t-t_d\right)}{\tau}\right\}\right] \tag{5.20-2}$$

or equivalently

$$y(t) = AKU(t-t_d)\left[1 - e^{-\xi(t-t_d)/\tau}\left(\frac{1-\xi^2+\left(\xi-\frac{M}{K\tau}\right)^2}{1-\xi^2}\right)^{1/2}\sin\left\{\sqrt{1-\xi^2}\,\frac{(t-t_d)}{\tau}+\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi-\frac{M}{K\tau}}\right\}\right]$$

(5.20-3)

5.21 unstable systems: $\xi < 0$

When the damping factor is less than zero, the exponential terms in (5.20-2) and (5.20-3) increase with time, and the system is unstable to disturbances.

5.22 inverse responses

When the disturbance rate-of-change factor M is non-zero, and of opposite sign to the gain K, the system can show an inverse response; that it, the initial response of the system is opposite to its ultimate direction. Figure 5.22-1 shows a second-order response that initially is negative, but ultimately oscillates around a positive change. The usual textbook example is that of level control in a boiler – adding water will initially lower the liquid level, as measured by the sensor, because bubbling is suppressed. Such inverse responses present challenges to controllers, because they start off by making the problem worse.

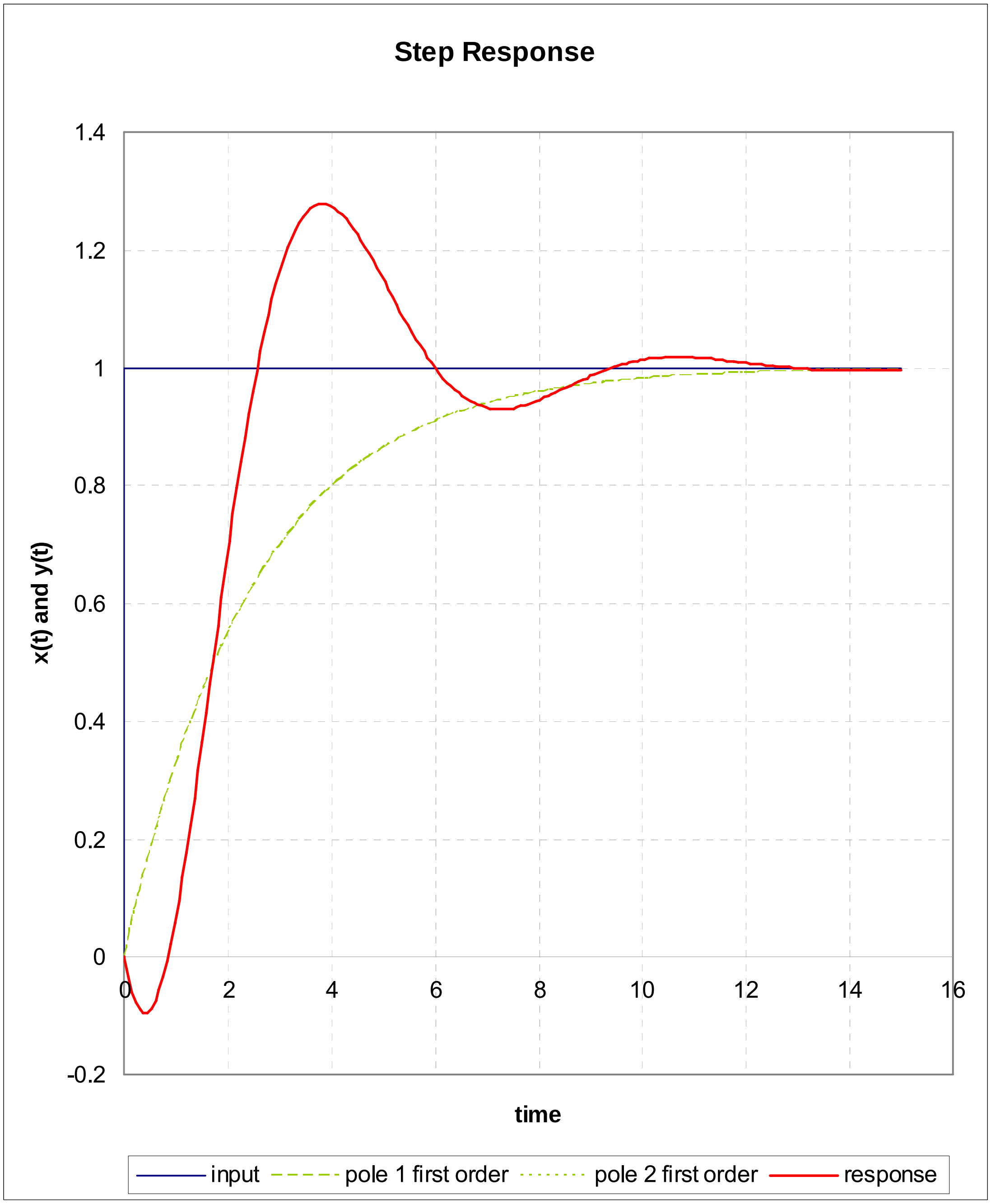


Figure 5.22-1. Second-order system; inverse response.

5.23 general map for second order

Figure 5.23-1 relates the qualitative behavior of a second-order system to its damping coefficient and $MK^{-1}\tau^{-1}$, a group that is the reciprocal of the product of the time constant and the transfer function zero. The major divisions of behavior correspond to the value of ξ, but various other features depend on this other group.

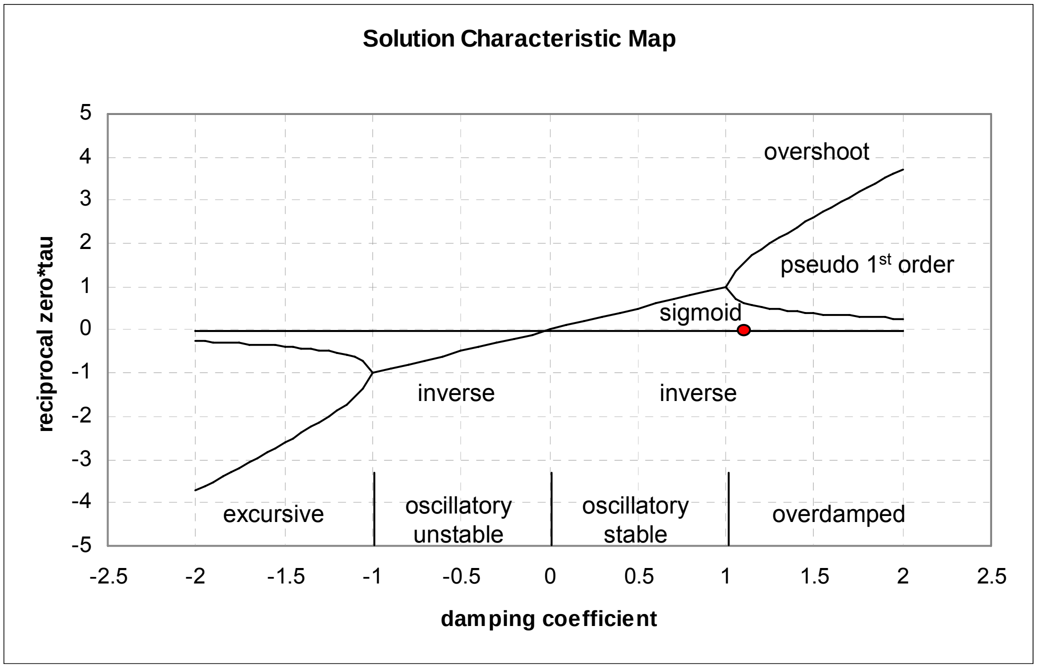


Figure 5.23-1. Map relating character of step response to model parameters

5.24 Measures that characterize the underdamped step response

Oscillatory responses are common. We have focused on second-order as a type for an oscillatory system: we have derived equations and inferred behavior. However, much process control work is conducted in the opposite direction: we observe oscillatory sensor readings, and we must try to infer a workable system model, as well as diagnose faults.

Practitioners use several measures to characterize an observed response (Coughanowr and Koppel, 1965). For the particular case of the 2nd order system, we can supplement these definitions with equations, but for an unknown system we can only compute the measures from recorded data.

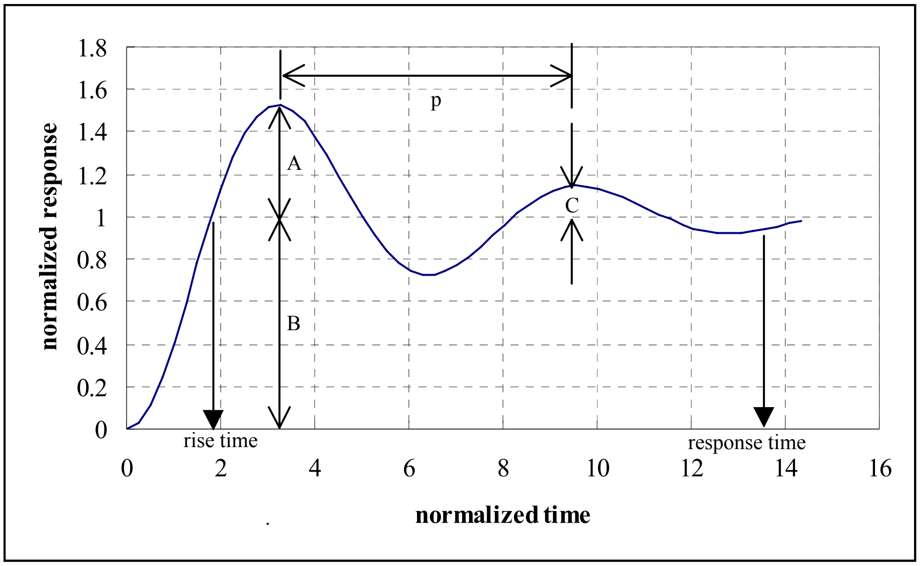


Figure 5.24-1. Oscillatory response to step input.

overshoot – response exceeds the ultimate value; equal to A/B

$$\text{overshoot} = e^{-\pi\xi / \sqrt{1-\xi^2}}$$

decay ratio – ratio of successive peaks, equal to C/A

$$\text{decay ratio} = \text{overshoot}^2 = e^{-2\pi\xi / \sqrt{1-\xi^2}}$$

rise time – time to first reach the ultimate value

response time – time until response remains within ±5% of ultimate value

period – time between peaks, or between alternate crossings of the ultimate value.

$$p\left(=\frac{1}{f}=\frac{2\pi}{\omega}\right)=\frac{2\pi\tau}{\sqrt{1-\xi^2}}$$

natural period – period if there is no damping. A step disturbance will cause an undamped system to oscillate perpetually about its ultimate value. Notice that the time constant τ is directly proportional to the natural period. Notice also that damping lengthens the period.

5.25 disturbance response for the heated tank

Armed with more knowledge of second order systems, we see that allowing the response to be underdamped will return the controlled variable to the vicinity of the set point more quickly than in an overdamped response. However, if the damping coefficient becomes too small, the oscillation amplitude and persistence may be unacceptable. We return to the disturbance transfer function in (5.15-2) and recognize the characteristic time and damping coefficient.

$$\tau_{cl} = \sqrt{\frac{\tau T_I}{K_s K_c K_v K_3}} \tag{5.25-1}$$

$$\xi = \frac{1 + K_s K_c K_v K_3}{2} \sqrt{\frac{T_I}{\tau K_s K_c K_v K_3}} \tag{5.25-2}$$

From (5.25-1) we see that increasing controller gain K_c and decreasing the integral time T_I tend to speed the loop response. Both these adjustments move the controller in the direction of aggressive tuning. The results of aggressive tuning are mixed on the damping coefficient - decreasing T_I increases oscillation, but increasing K_c suppresses it. The lower limit of ξ, however, is zero, so our second-order closed loop can be unstable (theoretically) only in the limit of zero integral time.

For a step disturbance in T1 of magnitude ΔT, we find from (5.15-2), (5.25-1), and (5.25-2)

$$T_o'(s) = \frac{\frac{K_1 T_I}{K_s K_c K_v K_3} s}{\tau_{cl}^2 s^2 + 2\tau_{cl}\xi s + 1} \frac{\Delta T}{s} \tag{5.25-3}$$

which may be inverted to give

$$T_o'(t) = \frac{K_1 T_I \Delta T}{K_s K_c K_v K_3} \frac{1}{\tau_{cl}\sqrt{1-\xi^2}} e^{-\xi t / \tau_{cl}} \sin \sqrt{1-\xi^2} \frac{t}{\tau_{cl}} \tag{5.25-4}$$

Equation (5.25-4) shows that the response oscillates about, and decays to, zero; as with the set point response we calculated in (5.16-2), there is no offset in the controlled variable, in spite of the permanent change in input. Thus integral-mode control has improved our ability to control the outlet temperature.

Figure 5.25-1 shows responses for several controller tunings – that is, several choices of parameters K_c (represented within the loop gain K) and T_I (scaled to the process time constant). Upon reducing the integral time, we reduce the amplitude of the error but undergo more oscillation. By increasing the gain, we speed the decay and thus reduce both the amplitude of the error and time spent away from set point: a 10 K inlet disturbance affects the outlet temperature by less than 1 K.

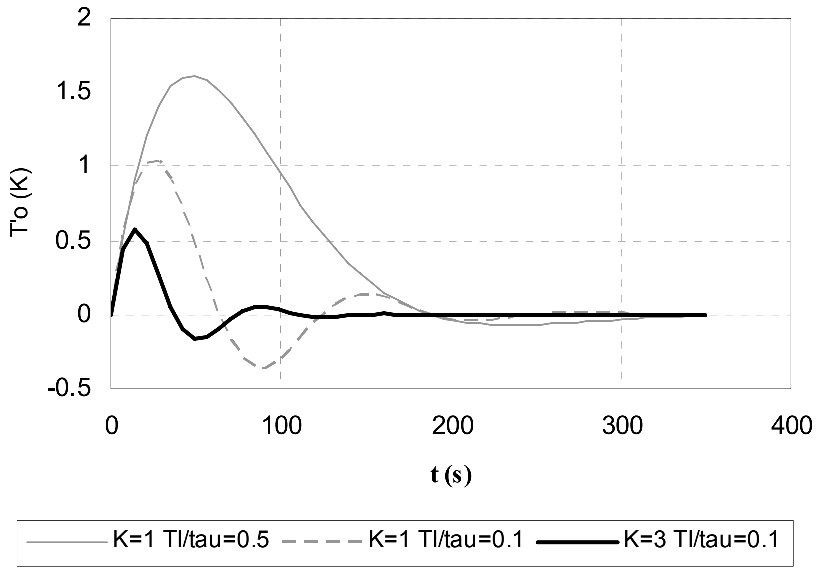


Figure 5.25-1. Step response of heated tank under PI control.

5.26 tuning the controller - measures of performance

In Lesson 4 we tuned by keeping our distance from the stability boundary, but did not consider whether performance was satisfactory. In Section 5.25 we saw that we could adjust two controller parameters independently to affect the response, and that there was a trade-off between amplitude and oscillation. It is time to introduce standard measures for the controlled variable that allow us to compare different tunings quantitatively.

integral error (IE) -- can be deceptively small if errors are balanced

$$\mathrm{IE} = \int_0^\infty \varepsilon(t)\,dt \tag{5.26-1}$$

integral absolute error (IAE) -- accounts for any deviation

$$\mathrm{IAE} = \int_0^\infty \left|\varepsilon(t)\right| dt \tag{5.26-2}$$

integral square error (ISE) -- emphasizes large errors

$$\mathrm{ISE} = \int_0^\infty [\varepsilon(t)]^2\,dt \qquad (5.26\text{-}3)$$

integral time absolute error (ITAE) -- emphasizes persistent errors

$$\mathrm{ITAE} = \int_0^\infty t|\varepsilon(t)|\,dt \qquad (5.26\text{-}4)$$

Of course, these integral error measures can be defined for scaled error ε^*, as well. The latter three will always increase as the controlled variable spends time away from the set point, so in general, smaller means better control. The measures can be calculated from plant data to compare the results of different tunings. They can also be used to compare the results of simulations. For example, we could calculate IAE for the three traces in Figure 5.25-1.

Furthermore, we could optimize by seeking a combination of tuning parameters that minimized one of the error measures. Thus, the controller would be tuned by an objective performance criterion. For response (5.25-4), of course, such an optimization would be expected to lead to infinite gain and zero integral time, because (as we determined by examining the poles of the transfer function in Section 5.16) the closed loop is stable for all tunings. In a less idealized system, however, an optimum is more likely to exist.

5.27 stability by Bode criterion

We were assured of closed loop stability in Section 5.16. Even so, we examine the frequency response of the loop transfer function.

$$G_L = \frac{K_s K_c K_v K_3}{\tau s + 1}\left(1 + \frac{1}{T_I s}\right) \qquad (5.26\text{-}1)$$

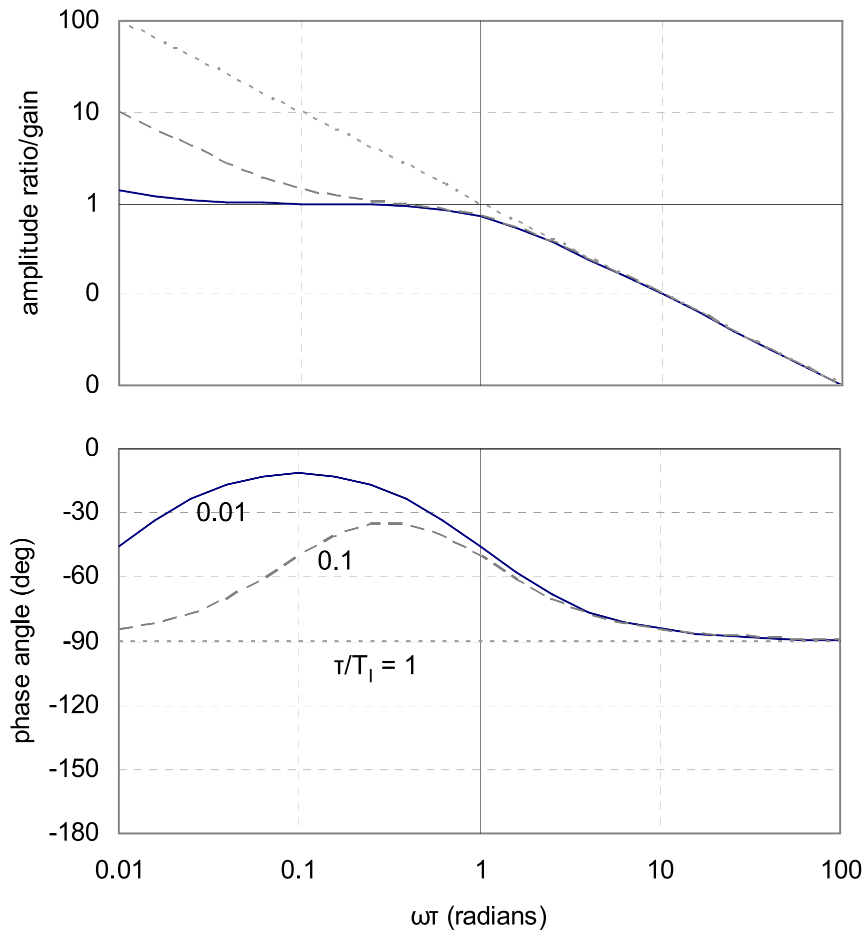
From the table of frequency response components in Marlin (Sec.10.6) we find

$$R_A = K_s K_c K_v K_3 \frac{\sqrt{1 + \frac{1}{T_I^2 \omega^2}}}{\sqrt{1 + \tau^2 \omega^2}} \qquad (5.26\text{-}2)$$

and

$$\phi = \tan^{-1}(-\omega\tau) + \tan^{-1}\left(\frac{-1}{\omega T_I}\right) \qquad (5.26\text{-}3)$$

The Bode plot is made for three ratios of tank time constant τ to controller setting T_I. For the least aggressive tuning with large T_I, the response resembles that of a first order system, although we notice different behavior at low frequencies, due to the integral controller. Decreasing the integral time depresses the phase angle toward a uniform -90º and increases the low-frequency amplitude ratio. The amplitude ratio on the plot is normalized by the loop gain; high controller gain settings would directly increase the amplitude of low-frequency disturbances.

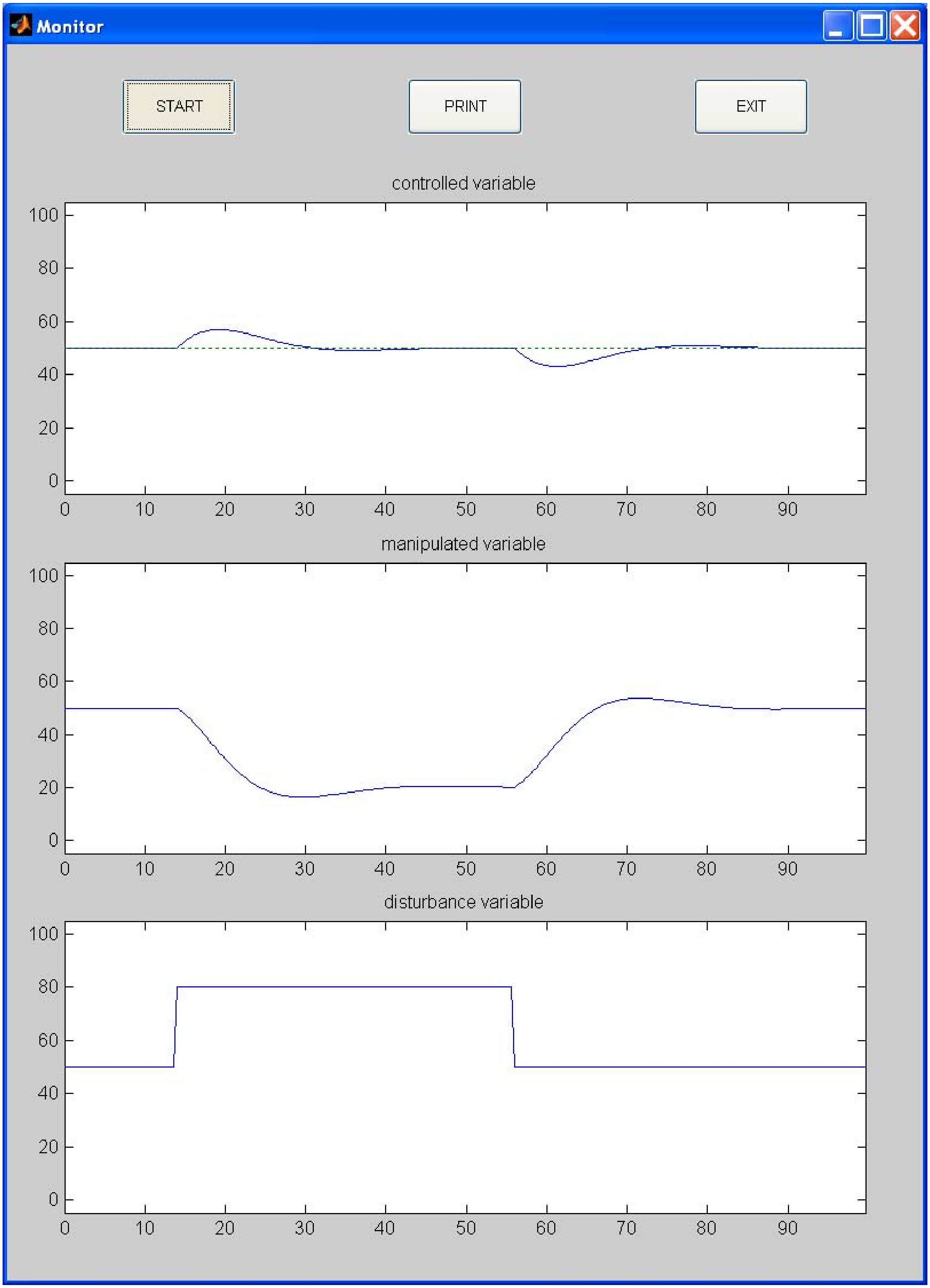


Because the phase lag never reaches -180º, the closed loop will remain stable. However, we see that the integral mode contributes phase lag at low frequencies, as well as boosts the amplitude ratio. Combined with other lags in a closed loop, integral control would tend to destabilize the loop.

5.28 numerical simulation of closed loop response

Numerical simulation of a first-order process under PI control was performed for a long pulse disturbance. The traces appear in scaled variables. The response is slightly underdamped.

Lesson 5: Heated Tank



Lesson 5: Heated Tank

5.29 conclusion

Integral control has been a big help by removing offset. However, it contributes phase lag to the closed loop, particularly at low frequencies, where it also boosts the amplitude ratio. Having two control parameters has allowed us more influence over the shape of the response.

5.30 references

Coughanowr, Donald R., and Lowell B. Koppel. *Process Systems Analysis and Control.* New York, NY: McGraw-Hill, 1965. ISBN: 0070132100.

Marlin, Thomas E. *Process Control.* 2nd ed. Boston, MA: McGraw-Hill, 2000. ISBN: 0070393621.