9.07 Introduction to Probability and Statistics for Brain and Cognitive Sciences
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Homework 5
October 19, 2016
Due October 26, 2016 at 5:00 PM

1. Let X be a binomial random variable with parameters n and p and let Y be a binomial random variable with parameters m and p. Assume that X and Y are independent. Show that $Z = X + Y$ is a binomial random variable with parameters $n + m$ and p. (Hint: Use the arguments for the sum of two Poisson random variables in **Example 5.4** in **Lecture 5** and the relationship $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{m+n}{k}$.)

2. In **Example 5.4**, we show that if X and Y are two independent Poisson random variables, with parameters λ_1, and λ_2 respectively, then $Z = X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$. Show that the pmf of X given Z is the binomial pmf with $n = z$ and $p = \lambda_1/(\lambda_1 + \lambda_2)$.

A. First, explain why

$$\Pr(X = x \cap Z = z) = \Pr(X = x \ \cap Y = z - x) = \frac{\lambda_1{}^x e^{-\lambda_1}}{x!} \frac{\lambda_2{}^{z-x} e^{-\lambda_2}}{(z-x)!}.$$

B. Next, to obtain the result compute

$$\Pr(X \mid Z) = \frac{\Pr(X = x \quad Z = z)}{\Pr(Z = z)} = \frac{\Pr(X = x \quad Y = z \quad x)}{\Pr(Z = z)}.$$

3. Suppose that X, Y and Z are independent discrete random variables and that each assumes the values 1, 2 and 3 with probability $\frac{1}{3}$.

A. Find the pmf of $W = X + Y$.

B. Find the pmf of $V = Z + W$.

4. The joint probability density (**Example 4.5, p.5** of **Lecture 5**) is difficult to visualize. Therefore, you want to simulate values from this density and make a scatter plot.

A. Assume that $\lambda = 3$ and use **Algorithm 5.1** to simulate in MATLAB® 500 draws (i.e. (X, Y) pairs) from $f_{xy}(x, y)$. This will entail first drawing X from the exponential density

$f_x(x) = 3e^{-3x}$,

using **Example 3.2**. Then, given $X = x$ draw Y from

$f_{y|x}(y \mid x) = 3e^{-3(y-x)}$,

again using **Example 3.2**.

B. Make a histogram plot of the X s. Does this look like what you would expect, i.e., the marginal density of X?

C. Make a histogram plot of the Y s. Does this look like what you would expect, i.e., the marginal density of Y?

5. The moment generating functions of the random variables in Problem 1 are for X, $\phi_x(t) = (pe^t + 1 - p)^n$ and for Y $\phi_y(t) = (pe^t + 1 - p)^m$. Solve Problem 1 using the moment generating functions, that is by finding the moment generating function of $Z = X + Y$.

6. Suppose X has a gamma distribution with parameters α and β. Using the moment generating function

A. Compute the skewness of X.

B. Compute the kurtosis of X.

7. The moment generating function of a Gaussian random variable is $\exp(\frac{\sigma^2 t^2}{2})$. Find its fourth moment.