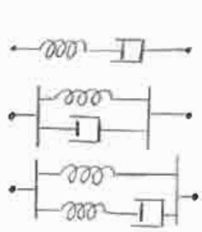
BEH. 410 . 04/02/
L # 14

Last time : - models of viscoelastic behavior (Maxwell, Voigt, SLS) , creep & stress relaxation
- oscillatory behavior

Today - applications of lumped models
- generalized LV descriptions
- continuum LV descriptions

▫ Lumped models

Maxwell
Voigt
SLS



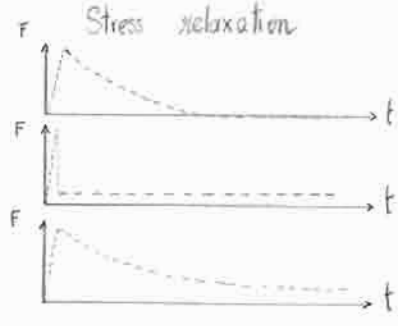
Generalized equations

$$\frac{du}{dt} = \frac{1}{k}\frac{dF}{dt} + \frac{F}{\eta}$$

$$F = \eta \frac{du}{dt} + k u$$

$$F + \alpha \frac{dF}{dt} = k_1 u + \beta \frac{du}{dt}$$

Stress relaxation



• data don't fit ⇒ add elements

▫ Oscillatory motion

$$u(t) = u_0 \cos(\omega t) = \mathcal{Re}\left[u_0 \exp(i\omega t)\right]$$

$$F(t) = F_0 \cos(\omega t + \phi) = \mathcal{Re}\left[F_0 \exp(i(\omega t + \phi))\right]$$

and

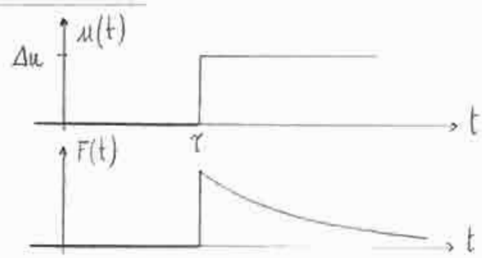
storage modulus $G'(\omega) = \frac{F_0(\omega)}{u_0}\cos\phi$

loss modulus $G''(\omega) = \frac{F_0(\omega)}{u_0}\sin\phi$

steady - state behavior

▫ Generalized linear viscoelastic models

Consider the experiment



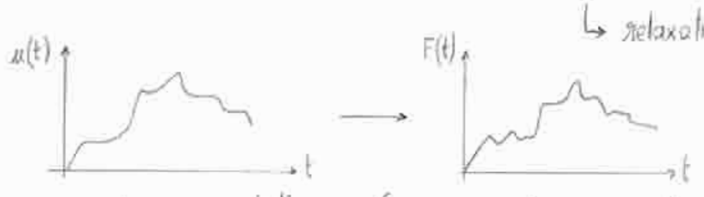
In general, let $F(t) = g(t-\tau)\,\Delta u(\tau)$

linear ⇒ superposition $F(t) = \sum_i g_i(t-\tau_i)\,\Delta u(\tau_i)$

limit of a smooth $u(t)$: $F(t) = \int_{-\infty}^{t} g(t-\tau)\,\frac{du(\tau)}{d\tau}\,d\tau$

↳ relaxation function (see chapter 1 - Ferry)

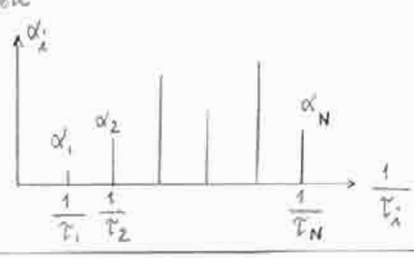
We want



general representation (many lumped elements) $g(t) = \sum_{i=0}^{N} \alpha_i \exp\left(-\frac{t}{\tau_i}\right)$

(recall for Maxwell $F(t) = k u_0 \exp\left(-\frac{t}{\tau_R}\right)$, $\tau_R = \eta/k$)

• discrete



or continuous

▹ Cells mechanics

single force & single displacement
but nonhomogeneous, anisotropic, complex composition & structure

▫ Lumped model → <u>continuum description</u>

lumped parameter $$F(t) = \int_{-\infty}^{t} g(t-\tau)\,\frac{du(\tau)}{d\tau}\,d\tau$$

continuum (see Ferry) $$\sigma_{ij}(\bar{x},t) = 2\int_{-\infty}^{t} G_{ijkl}(\bar{x},t-\tau)\,\frac{d\varepsilon_{kl}}{d\tau}\,d\tau$$

↳ tensorial relaxation function

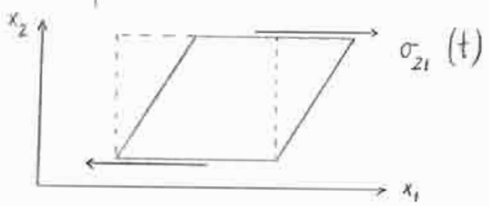
and for viscoelastic fluid $$\sigma_{ij} = 2\int_{-\infty}^{t} \frac{dG_{ijkl}(\bar{x},t-\tau)}{d\tau}\,\varepsilon_{kl}(t,\tau)\,d\tau$$

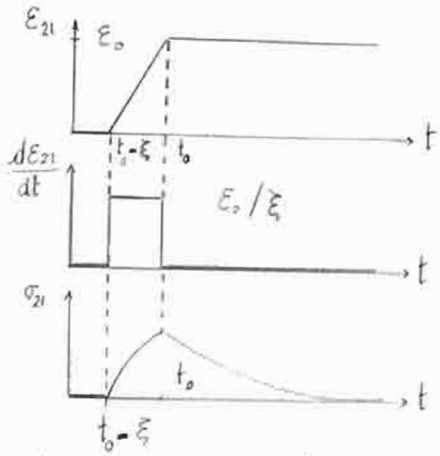
↳ strain at t relative to strain at τ

What is the relationship between G_{ijkl} and G ?

↳ shear modulus

. linear pure shear deformation





$$\sigma_{21}(t) = 2\int_{-\infty}^{t} G(t-\tau)\,\frac{d\varepsilon_{21}(\tau)}{d\tau}\,d\tau$$

$$= 2\int_{t_0-\xi}^{t_0} G(t-\tau)\,\frac{\varepsilon_0}{\xi}\,d\tau \quad \text{for } t \gg t_0$$

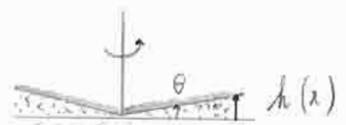
$$= 2\,G\big(t-(t_0+\xi\delta)\big)\,\frac{\varepsilon_0}{\xi}\,\xi \qquad , 0\leq\delta\leq 1$$

for $\xi \ll t$ and $t_0 = 0$, we have $\boxed{\sigma_{21}(t) = 2\,G(t)\,\varepsilon_0}$ linear viscoelastic solid

$\sigma_{21} = 2G\ \varepsilon_{21}$ linear elastic solid

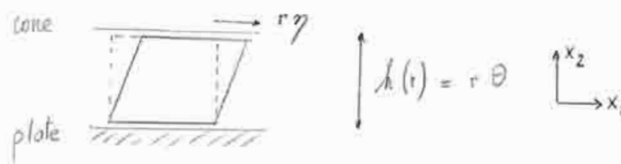
$G(t)$ not to be confused with $G'(\omega)$ or $G''(\omega)$

. ex. <u>cone-plate rheometer</u>



top view

side view



impose oscillatory shear $\varepsilon_{21} = \frac{1}{2}\,\frac{\partial u_1}{\partial x_2} = \frac{1}{2}\cdot\frac{r\eta}{r\theta} = \frac{\eta}{2\theta}$ independent of radius

measure torque $T = \int_0^R \sigma_{21}\ r\ 2\pi r\ dr = 2\pi\,\sigma_{21}\,\frac{R^3}{3}$

$$\sigma_{21}(t) = \frac{3}{2\pi}\cdot\frac{T}{R^3}$$ because T measurable, $\sigma_{21}(t)$ is, too

recall $\hat{G}(\omega) = \dfrac{\hat{F}(\omega)}{u_0} = G' + i\,G''$

or continuum $\hat{G}(\omega) = \dfrac{\hat{\sigma}_{21}(\omega)}{2\hat{\varepsilon}_{21}} = \dfrac{\sigma_0 \exp(i\phi)}{2\varepsilon_0}$ because $\hat{\sigma}_{21} = \sigma_0 \exp(i(\omega t + \phi))$

$\hat{\varepsilon}_{21} = \varepsilon_0 \exp(i\omega t)$

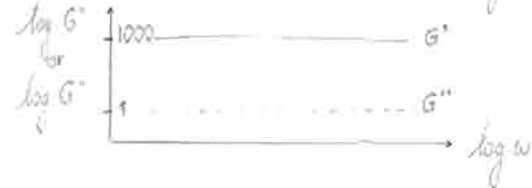
$$\hat{G}(\omega) = \frac{\sigma_0}{2\varepsilon_0}\left(\cos\phi + i \sin\phi\right)$$

$$= \underbrace{\frac{\sigma_0}{2\varepsilon_0}\cos\phi}_{\to G'} + i \underbrace{\frac{\sigma_0}{2\varepsilon_0}\sin\phi}_{\to G''}$$

- look at before gelation

viscous liquid

after gelation



elastic

- limiting cases

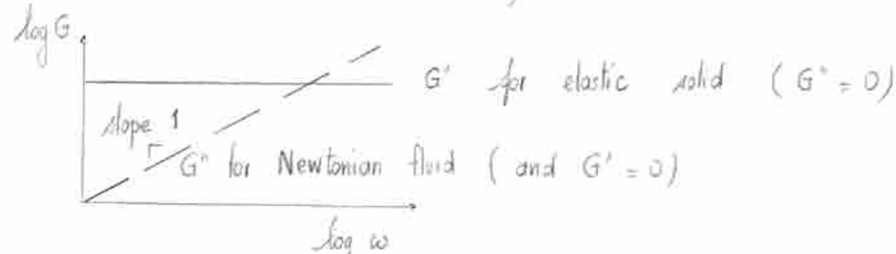
$$\hat{G}(\omega) = \frac{\sigma_0}{2\varepsilon_0}\cos\phi + i\frac{\sigma_0}{2\varepsilon_0}\sin\phi$$

elastic solid $\phi \to 0$ $\hat{G} = \dfrac{\sigma_0}{2\varepsilon_0} = G'$ real and $G'' = 0$

viscous liquid $\phi \to \dfrac{\pi}{2}$ $\hat{G} = i\dfrac{\sigma_0}{2\varepsilon_0} = iG'$ pure imaginary and $G' = 0$

Newtonian fluid $\sigma_{21} = 2\mu \dfrac{d\varepsilon_{21}}{dt} = -2\underbrace{\mu\,\varepsilon_0\,\omega}_{\sigma_0}\sin(\omega t)$ and $G'' = \dfrac{2\mu\varepsilon_0\omega}{2\varepsilon_0}$

$$G''(\omega) = \mu\omega$$



. Oscillating sphere inside an infinite viscoelastic medium (Zieman et al 1984, Sackmann group)

optical trap → $u(t)$, magnetic trap → $F(t)$

bead in medium $\begin{cases}\text{isotropic}\\ \text{homogeneous}\\ \text{linearly viscoelastic}\end{cases}$

$u = u_0 \cos(\omega t)$

$F = F_0 \cos(\omega t + \phi)$

superimpose F_V & F_E defined as follows

- if $\begin{cases}\text{purely viscous}\\ \text{Newtonian}\\ \text{no inertia}\end{cases}$ Stokes flow $F_V(t) = 6\pi\mu a \dfrac{du}{dt}$

μ ↑ coefficient of viscosity

- if purely elastic $F_E(t) = 6\pi G a u$